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On the Nonlinear Deformation Geometry of Euler-Bernoulli Beams

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Dewey H. Hodges, Robert A. Ormiston,
and David A. Peters

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On the Nonlinear Deformation Geometry of Euler-Bernoulli Beams

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SYMBOLS

B	rigid body located at point P in which the vectors **i**, **j**, **k** are fixed
$c_{(\ )}$	cosine ( )
GJ	beam torsional rigidity, N-m^2
H	kinetic energy minus potential energy, N-m
i, **j**, **k**	unit vectors fixed in body B that, for the initial position of B, lie along the axes x,y,z and for the final position of B, the deformed state, lie along the axes x_3, y_3, z_3
I, **J**, **K**	unit vectors along the reference axes x,y,z (fig. 1)
L	beam length, m
L_x, L_y, L_z	distributed applied loads in the x,y,z directions, respectively, N/m
M_{x_3}	distributed applied moment about the x_3 axis (i.e., an applied pitching moment), N-m/m
$O(\varepsilon^3)$	a quantity that has order of magnitude less than or equal to third degree in the rotations
P	a generic point on the beam elastic axis
r	the distance along the deformed-beam elastic axis measured from the beam root, m
$s_{(\ )}$	sine ( )
[T]	the transformation matrix of direction cosines relating the axes x_3, y_3, z_3 to x, y, z
t	time, sec
t_1, t_2	arbitrary times, sec
u	deflection of point P in the x direction, referred to as axial deflection, m
u_e	axial deflection of point P due to longitudinal strain of the elastic axis, a spatial quasi-coordinate defined as $u + (1/2)\int_0^x (v'^2 + w'^2)dx$, m
v	deflection of point P in the y direction, referred to as lead-lag deflection, m

w	deflection of point P in the z direction, referred to as flap deflection, m
x, y, z	a set of Cartesian coordinates with origin at the beam root, the x axis being along the undeformed beam, m (fig. 1)
x_3, y_3, z_3	a set of Cartesian coordinates with origin at point P with x_3 remaining tangent to the elastic axis and y_3 and z_3 along principal axes for the cross section in the deformed state, m
β	a rotation (flap) of the body B about the unit vector $-\mathbf{j}$; the exact axis about which the rotation occurs is dependent on the sequence of angular rotations, rad
β_0	angle with the $x - y$ plane of the beam in the example problem, rad
δ	variational operator; also Dirac delta function
$\underline{\delta W}$	virtual work per unit length done by nonconservative applied loads, N-m/m
$\boldsymbol{\delta\psi}$	vector of virtual rotations of the reference frame x_3, y_3, z_3 fixed in the beam cross section, rad
ζ	a rotation (lead-lag) of the body B about the unit vector $\mathbf{k}$; the exact axis about which the rotation occurs is dependent on the sequence of rotations, rad
θ	a rotation (pitch) of the body B about the unit vector $\mathbf{i}$; the axis about which the rotation occurs is dependent on the sequence of rotations, rad
θ_t	pretwist angle, rad
θ_1	the third angle in the series lag-flap-pitch; it is the third angle required to rotate body B to the final position after first lag then flap rotations, rad
θ_2	the third angle in the series flap-lag-pitch; it is the third angle required to rotate body B to the final position after first flap then lag rotations, rad
κ_0	the $\mathbf{k}$ component of curvature of the beam in the example problem, m^{-1}
$\boldsymbol{\kappa}$	rotation per-unit-length vector of the reference frame x_3, y_3, z_3; the $\mathbf{i}$ component is the torsion (angle of twist per unit length) and the $\mathbf{j}$ and $\mathbf{k}$ components are bending curvatures, m^{-1}

ϕ	angle of twist due to torsional shear strain, a spatial quasi-coordinate defined as $\int_0^r \kappa_i \, dr - \theta_t$, rad
ω	angular velocity of the reference frame x_3, y_3, z_3, rad/sec

Subscripts and Superscripts

1	lag-flap-pitch sequence of rotations
2	flap-lag-pitch sequence of rotations
3	lag-pitch-flap sequence of rotations
4	flap-pitch-lag sequence of rotations
5	pitch-lag-flap sequence of rotations
6	pitch-flap-lag sequence of rotations
$(\dot{})$	$\partial/\partial t(\ )$
$(\ )^+$	$\partial/\partial r(\ )$
$(\ )'$	$\partial/\partial x(\ )$
$(\ )_{i,j,k}$	(vector) $\cdot$ **i**, **j**, **k**
$(\ )_{I,J,K}$	(vector) $\cdot$ **I**, **J**, **K**

ON THE NONLINEAR DEFORMATION GEOMETRY OF EULER-BERNOULLI BEAMS

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SUMMARY

Nonlinear expressions are developed to relate the orientation of the deformed-beam cross section, torsion, local components of bending curvature, angular velocity, and virtual rotation to deformation variables. These expressions are developed in an exact manner in terms of a quasi-coordinate in the space domain for the torsion variable. The entire formulation is independent of the sequence of the three rotations used to describe the orientation of the deformed-beam cross section. For more common cases in the literature in which one of the three rotation angles is used as the torsion variable, the resulting equations depend on the choice of the three angles. Differences in the equations, however, are demonstrated to be in form only. The present deformed-beam kinematic quantities are proven to be equivalent to those derived from various rotation sequences by identifying appropriate changes of variable based on fundamental uniqueness properties of the deformed-beam geometry. This development helps to clarify the issues raised in the literature concerning the choice of the angles. The torsion variable used herein is also shown to be mathematically analogous to an axial deflection variable that has been commonly used in the literature. Both variables are quasi-coordinates in the space domain and have been used in derivations based on Hamilton's principle, despite lack of rigorous justification. Rigorous applicability of Hamilton's principle to systems described by a class of quasi-coordinates that includes these variables is formally established.

1. INTRODUCTION

1.1 Background

To adequately model a helicopter blade, the deflections must be treated as moderately large and the equations of motion will be nonlinear. Such equations are required for properly analyzing the stability and forced response of helicopter rotor blades in hover and in forward flight. To derive such equations, it is necessary to first specify the geometry of the beam both in its undeformed state and in its deformed state at some particular instant in time. For typical beam theories, this involves expressing the position of a generic

point P on the elastic axis, the orientation of a frame consisting of the axes normal to and along principal axes for the cross section at P, and any further deformations, such as warp of the cross section out of its nominal plane, to adequately specify the location of every material point in the beam (ref. 1). In the discussion of kinematics that follows we will focus on the variables used to describe the location of point P on the elastic axis and the orientation of the reference axes attached to the beam cross section at P, assuming that no deformation of the beam cross section occurs. This would, in general, imply a total of six variables: three deflection variables for the location of P and three angles[1] for the orientation of the reference axes with origin at P with respect to a suitable space-fixed reference frame. For an Euler-Bernoulli beam, however, the cross section is assumed to remain normal to the beam elastic axis during deformation. Thus, two of the three angles can be eliminated by expressing them in terms of derivatives of the deflection variables. It is then necessary to express all the kinematical quantities in terms of the four remaining variables.

It is clear that a material point can only occupy one position at a time. The beam geometry, by which we refer collectively to the position of the material points along the elastic axis and the orientation of the cross sections in terms of direction cosines for the reference axes at P, is then unique at a specific time. When the beam geometry is specified, so is the strain, the torsion (or angle of twist per unit length), and the components of bending curvature along the principal axes of the cross section at P, since these also are geometric quantities. Although the values of these quantities are unique at a particular instant in time, the mathematical variables used in defining the geometry are not unique. In reference 2, Reissner shows, for example, that it is possible to formulate a large-deflection beam theory completely independent of the choice of deformation variables. These equations are an extension of the Kirchhoff-Love equations (ref. 1) and illustrate the obvious fact that the geometric quantities describing the beam deformation under a given load, although the expressions may look different for different choices of variables, are determined by the loads alone. There are many possibilities, for example, in the choice of angles used to relate the orientation of the reference axes at P to a space-fixed Cartesian frame (fig. 1). In general, any change in orientation can be described by three successive rotations about three distinct, well-defined axes. These sequential rotations can either occur about space-fixed axes, or about axes that are fixed in an updated reference frame resulting from a previous rotation. The angles themselves are not unique, but values of the resulting direction cosines that define the orientation are unique.

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[1]It is possible to use variables other than angles to describe changes in orientation such as Rodriguez parameters (ref. 2). In this paper, however, we will restrict the discussion to the use of angles.

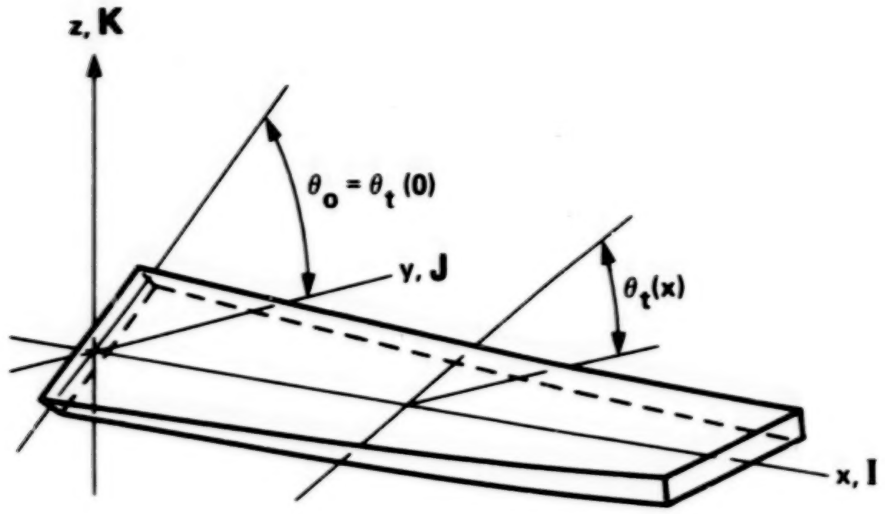

Figure 1.- Undeformed, pretwisted beam and reference Cartesian axes x, y, z.

1.2 Discussion of Previous Work

With the assumption that the beam cross section remains normal to the elastic axis during deformation, the exact expressions describing the deformed beam geometry have been developed and applied to a derivation of the equations of motion for a helicopter blade (refs. 3-5). In reference 3, the exact direction cosines for the reference axes with origin at P were derived. These direction cosines are invariant with the choice of angles used in describing the orientation. The direction cosines constitute elements of a transformation matrix [T] relating the reference axes with origin at P and a space-fixed set of Cartesian axes, the x axis of which is along the undeformed beam (fig. 1).

In reference 4, the structural and inertial operators were derived, through second-degree nonlinearity, for the rotating beam. The bending curvatures were developed in the appendix of reference 4. These expressions are exact and independent of the choice of angles used to describe the orientation. In reference 5, expressions for the aerodynamic loads are developed for the rotating beam. To derive expressions for the aerodynamic loads one must make use of components of each of the following vector quantities along the reference axes with origin at P: the relative velocity of point P with respect to the fluid, the angular velocity of a rigid body B in which are fixed the reference axes with origin at P, and the virtual rotation of B (fig. 2). The first of these three items depends only on the [T] matrix, but the latter two must be developed separately. Only approximate forms of these quantities are given in reference 5 and the exact expressions, developed herein, have not previously been published.

Torsion deformation in references 3-5 is described by use of a quasi-coordinate in the space domain, the elastic component of the angle of twist. The variable is referred to as a quasi-coordinate because it is related to

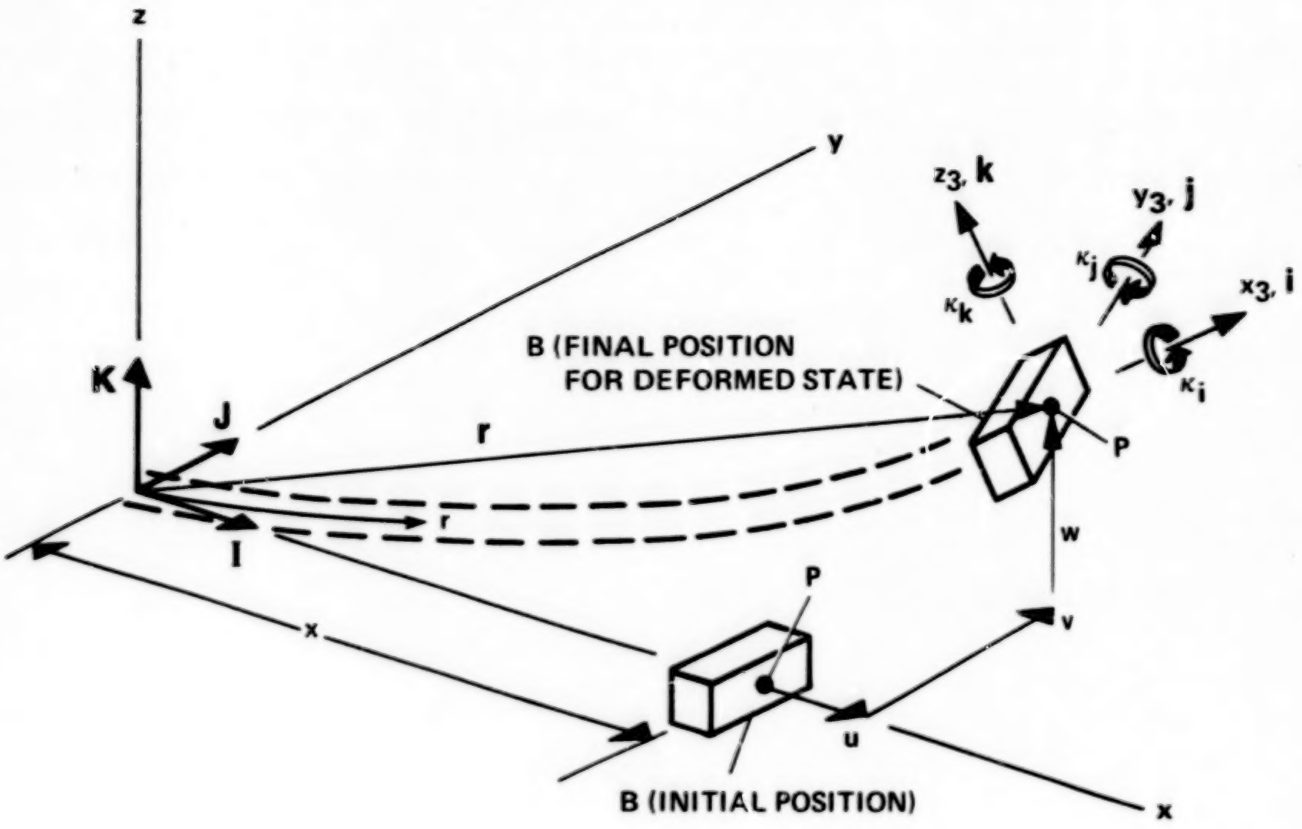

Figure 2.- Deformed-beam geometry.

angles through integrals that cannot be evaluated in closed form. Hence, all three angles, no matter how they are chosen, can be eliminated from the derivation for an Euler-Bernoulli beam. Although the definition of the torsion variable is mathematically explicit in references 3-5, it is never stated therein that it is a quasi-coordinate. This may account for some of the discussions that have appeared in the literature (refs. 6-8).

To better understand these discussions it is helpful to consider a simple approximation. Sometimes an elastic beam is approximated by a system of spring-restrained rigid bodies, the simplest of which is a single rigid body, hinged at a point corresponding to the beam root to provide three angular degrees of freedom. Now, for a rigid-body system such as this, the arrangement of hinges presents the analyst with a choice of infinitely many distinct physical spring arrangements — each of which constitutes a distinct, well-defined physical system. For example, in reference 3 differences in stability between two of these systems are identified. They each have only two degrees of freedom: flap (out of the plane of rotation) and lead-lag (nominally in the plane of rotation). The only difference between the two systems is the hinge arrangement. By definition, lag-flap means that the flap hinge leads or lags with the blade; flap-lag means that the lead-lag hinge flaps with the blade. Thus, lead-lag motion is precisely inplane only for the lag-flap arrangement; hence, the word nominally is used in the above definition. For small angles, the equations are linear and the two systems have identical equations of

motion. Any degree of nonlinearity produced by moderately large angles will, however, produce differences in the blade pitch orientation for the two systems for given flap and lead-lag deflections. These differences, although not large, produce different dynamic behavior and stability for the two systems. It is clear, however, that an elastic beam with a particular set of end constraints is only one physical system and its kinematics, stability, and dynamic behavior are not subject to the analyst's choice of angles used in the derivation of equations that describe the system. While this may seem clear, there has been considerable discussion and some misunderstanding of this issue in the literature (refs. 6-8).

The role of hinge arrangement in determining coupled flap-lag stability of a rotating, centrally hinged, rigid-beam model of a helicopter rotor blade in hover and in forward flight is further addressed in reference 6 where both lag-flap and flap-lag systems are studied. The arrangement of the hinges is shown to influence the stability boundaries for both hover and forward flight. In reference 7 two different sequences of rotations are used to describe the orientation of the cross section of the deformed elastic beam and second-degree nonlinear expressions are developed for torsion, the bending curvatures, and the [T] matrix. These expressions are used in reference 8 to derive two sets of equations of motion for a rotating, elastic-beam model of a helicopter blade in hover and in forward flight. In the two sets of equations, the torsion deformation is described by the appropriate angles of the two sets, which differ from each other and from the variable of references 3-5 as well. The differences in the variables are not mentioned in references 7 and 8, and in reference 8 the fact that the formulations must be equivalent is only conceded to be an unproven possibility. The following conclusions are drawn in references 6-8: (1) the transformation sequence used in deriving the equations influences the stability and dynamic behavior of an elastic beam (refs. 6, 8); and (2) the work of references 3-5 is incorrect in some respects (refs. 7, 8).

Because of the importance of the fundamental geometric relationships of slender beams to the nonlinear analysis of rotor blades, and because of the apparent differences that exist between references 3-5 and references 6-8, a detailed exposition of this subject has been undertaken in this report. The influence of the choice of angles is thoroughly investigated in this report. The fact that any of the hinged-beam approximations can be analyzed with any choice of angles will be demonstrated. Although the hinge arrangement is crucial to accurate analysis, the choice of angles used in modeling is immaterial (ref. 9) as long as singularities, discussed in section 2, are avoided (ref. 10). Since an elastic beam, unlike the rigid beam with its many possible hinge arrangements, is a single physical system, we should not expect the choice of angles to influence the outcome of the analysis at all. Otherwise a completely intrinsic formulation like the Kirchhoff-Love equations (ref. 1) or their recent extension by Reissner (ref. 2) would be impossible. Finally, the work in references 3-5, while essentially correct, does suffer from lack of clarity in some areas and it is hoped that this report will clarify the earlier work as well as the ensuing discussions in references 6-8. Part of the difficulty in reconciling the results of other investigations with references 3-5 stems from the nature of the torsion variable used therein. The

properties of this variable need to be clarified, especially with respect to its use in conjunction with Hamilton's principle.

1.3 Procedure

In this report, the appropriate relations describing the geometry of the deformed beam are rederived and expanded to include the components of angular velocity and virtual rotation. The appropriate changes of variable are established to prove that the two formulations in references 7 and 8 are equivalent to each other and equivalent to the approximate form of the exact expressions derived herein and in references 3-5. A thorough comparison with the results of other work is undertaken along with a discussion of the geometric nature of the different torsion variables in references 3-8. The consistent reductions of the kinematical equations for infinite torsional rigidity and for the hinged, rigid-beam approximations are also examined. Finally, the application of Hamilton's principle to systems described by the torsion variable of references 3-5 is rigorously established.

2. THE LARGE-DEFLECTION GEOMETRY OF AN EULER-BERNOULLI BEAM

In this section the large-deflection geometry of an Euler-Bernoulli beam is derived. For the purpose of discussing the geometry, we will not consider warp. Without warp, cross sections remain plane and normal to the deformed-beam elastic axis in an Euler-Bernoulli beam. Thus, the geometry is completely determined by three deflections u, v, w along the x, y, and z axes (fig. 2) and some appropriate measure of the elastic twist. We must then express all other geometric quantities in terms of these four deformation variables. The orientation of the reference axes with origin at P with respect to the space-fixed axes is given by elements of the [T] matrix. Also, expressions for the torsion and components of bending curvature, angular velocity, and virtual rotations are developed.

We consider an initially straight beam segment with the elastic axis along the x axis, illustrated in figure 1. In its undeformed state the beam is pretwisted so that the principal axes of the cross section, denoted by y_3 and z_3, are at an angle $\theta_t(x)$ with the y and z axes, respectively, and $\theta_t(0)$ is denoted by θ_0. The cantilever root boundary condition is considered here for the purpose of illustration. In principle, any set of boundary conditions could be treated by slightly modifying the method that follows.

When the beam bends and twists to some deformed configuration, as in figure 2, the geometry can be completely described by the deflections u, v, w of the point P on the elastic axis and by the direction cosines of the reference axes x_3, y_3, and z_3 where x_3 is tangent to the elastic axis of the deformed beam. The matrix of direction cosines of x_3, y_3, z_3 with respect to x, y, z constitutes a coordinate transformation matrix denoted by [T] between the unit vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ associated with the local reference axes x_3, y_3, z_3 of the deformed beam and the unit vectors $\mathbf{I}$, $\mathbf{J}$, $\mathbf{K}$ associated with the space-fixed axes x, y, z. The relationship is given by

$$\begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix} = [T] \begin{Bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{Bmatrix} \tag{1}$$

When any two of the axes about which rotations occur have the possibility of being coincident within the primary range of interest of the rotations, singularities result. This is the disadvantage in using what are commonly known as classical Euler angles (ref. 10) since the first and third rotations are about the same axis when the second angle is zero. The remedy is to have the rotations about axes that do not approach one another for rotations in the neighborhood of zero since small deformations are a special case of interest for the present problem. These modified angles, as used in this development, are also sometimes called Euler angles in the literature (e.g., ref. 11). We assume that the unit vectors **i**, **j**, **k** are fixed in a rigid body B and are initially coincident with axes x, y, z. Rotations ζ, β, and θ of B occur about **k**, **-j**, and **i**, respectively, but not necessarily in that order, so that **i**, **j**, **k** are finally aligned with the x_3, y_3, z_3 axes. Therefore, the angular displacement θ rotates the beam cross section about what is, for $\zeta = \beta = 0$, the elastic axis; the angular displacement β rotates the beam cross section out of what is, for $\zeta = \theta = 0$, the horizontal (x-y) plane; the angular displacement ζ rotates the beam cross section in what is, for $\beta = \theta = 0$, the horizontal plane. The first rotation always occurs about either x, y, or z. The orientation of the exact axis about which the second rotation occurs is dependent on the axis and magnitude of the first rotation. The third rotation always occurs about either x_3, y_3, or z_3. In the text of this report, only two of the six possible sequences of angles will be considered: ζ_1, β_1, θ_1, and β_2, ζ_2, θ_2. The subscripts are added to emphasize the fact that, for a given orientation of x_3, y_3, z_3 with respect to x, y, z, $\zeta_2 \neq \zeta_1$, $\beta_2 \neq \beta_1$, and $\theta_2 \neq \theta_1$. Since it is clear that the choice of angles will not change the outcome of an analysis, it is natural to inquire at this stage why it is necessary to consider two different sequences. It is one purpose of this report to study the outcome of expressing the deformation kinematics in terms of the two sequences and to address the issues listed in section 1.2.

The axes of rotation for the ζ_1, β_1, θ_1 sequence may be described by a rotation ζ_1 about the z axis resulting in a new set of axes x_1^1, y_1^1, z_1^1, followed by a rotation β_1 about the $-y_1^1$ axis resulting in a new set of axes x_2^1, y_2^1, z_2^1, finally followed by a rotation θ_1 about the x_2^1 axis resulting in the x_3, y_3, z_3 axes (see fig. 3). The axes of rotation for the β_2, ζ_2, θ_2 sequence may be described by a rotation β_2 about the -y axis resulting in a new set of axes x_1^2, y_1^2, z_1^2, followed by a rotation ζ_2 about the z_1^2 axis resulting in a new set of axes x_2^2, y_2^2, z_2^2, finally followed by rotation θ_2 about the x_2^2 axis resulting in the x_3, y_3, z_3 axes (see fig. 4). The final set of axes must be the same regardless of the sequence of rotations. The matrix [T] may be formulated in terms of any set of three angles that will rotate the axes x, y, z to x_3, y_3, z_3. The discussion here is limited to the sequences ζ_1, β_1, θ_1 and β_2, ζ_2, θ_2. However, all six possible sequences of the angles ζ, β, and θ, as defined above, are used to develop the geometry in the appendix of this report. The conclusions hold

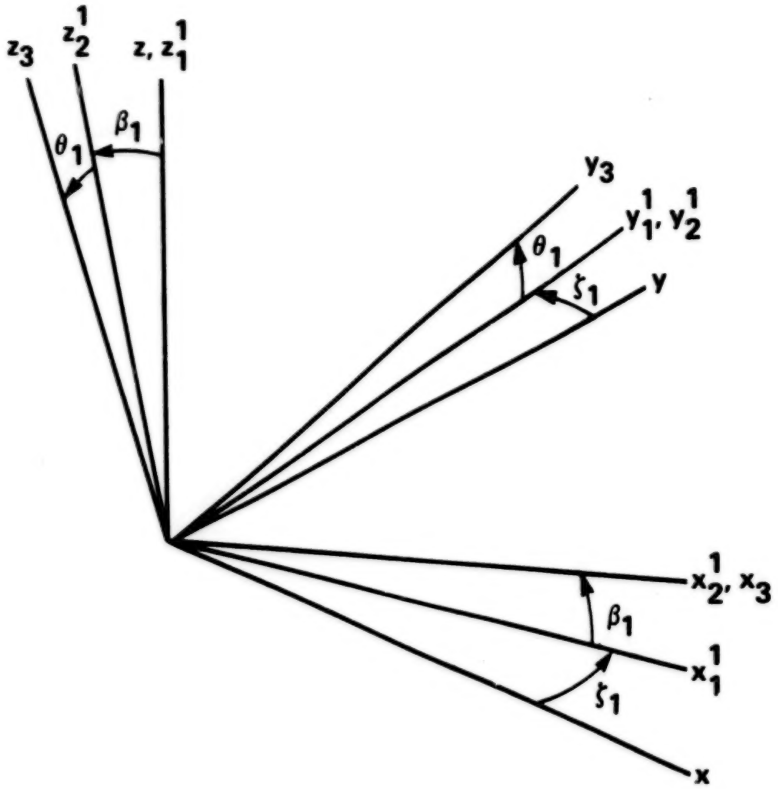

Figure 3.- The angles ζ_1, β_1, θ_1 and respective axes of rotation.

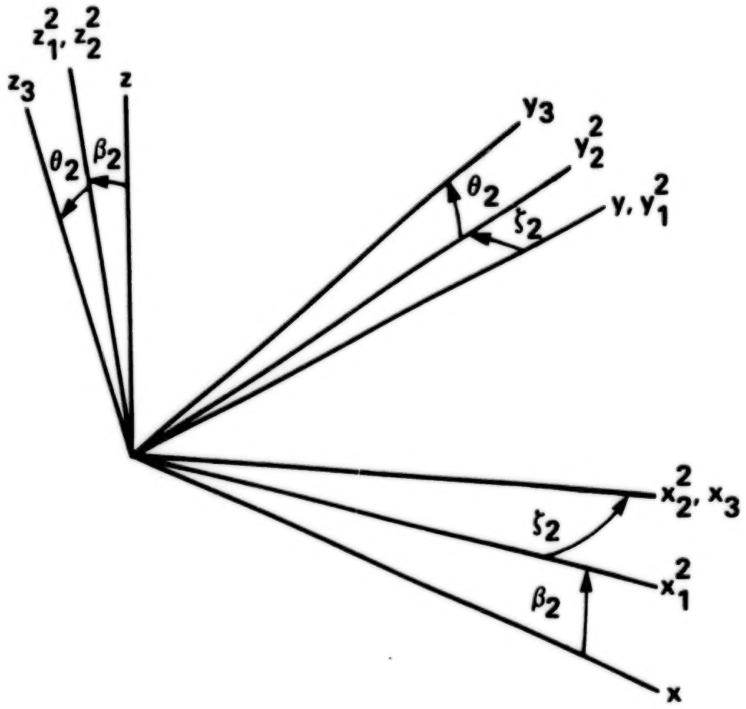

Figure 4.- The angles β_2, ζ_2, θ_2 and respective axes of rotation.

for any set of angles, regardless of definition, as long as they will describe the orientation of axes x_3, y_3, z_3.

The geometry of the deformed state is now developed. The position vector describing any point on the deformed-beam elastic axis can be written as (see fig. 2)

$$\mathbf{r} = (x + u)\mathbf{I} + v\mathbf{J} + w\mathbf{K} \tag{2}$$

The unit vector tangent to the elastic axis of the deformed beam is

$$\frac{\partial \mathbf{r}}{\partial r} = (x + u)^{+}\mathbf{I} + v^{+}\mathbf{J} + w^{+}\mathbf{K} \tag{3}$$

where r is the curvilinear distance coordinate along the deformed-beam elastic axis and $(\ )^{+} = \partial/\partial r(\ )$. Since the cross section remains normal to the elastic axis during deformation

$$\frac{\partial \mathbf{r}}{\partial r} = \mathbf{i} = T_{11}\mathbf{I} + T_{12}\mathbf{J} + T_{13}\mathbf{K} \tag{4}$$

where T_{ij} is the element on the ith row and in the jth column of [T]. Thus

$$\left.\begin{aligned} T_{11} &= (x + u)^{+} \\ T_{12} &= v^{+} \\ T_{13} &= w^{+} \end{aligned}\right\} \tag{5}$$

no matter what angles have been used to express [T]. Since the [T] matrix is orthonormal

$$T_{11}^2 + T_{12}^2 + T_{13}^2 = 1 \tag{6}$$

and a relationship between x and r is easily obtained from equations (5) and (6)

$$(x + u)^{+} = \sqrt{1 - v^{+2} - w^{+2}} \tag{7}$$

or

$$r' = \sqrt{(1 + u')^2 + v'^2 + w'^2} \tag{8}$$

where $(\ )' = \partial/\partial x(\ )$. The preceding equations must be combined with explicit expressions for the elements of [T] to completely specify the geometry of the deformed beam.

Explicit expressions for the [T] matrix, components of the angular velocity and virtual rotation vectors, the torsion and components of bending curvature are now developed. The ζ_1, β_1, θ_1 sequence is chosen arbitrarily for this task. The [T] matrix can be determined from the successive transformations relating x_3, y_3, z_3 to x, y, z using figure 3:

$$[T] = \begin{bmatrix} c_{\beta_1} c_{\zeta_1} & c_{\beta_1} s_{\zeta_1} & s_{\beta_1} \\ -c_{\zeta_1} s_{\beta_1} s_{\theta_1} - c_{\theta_1} s_{\zeta_1} & c_{\zeta_1} c_{\theta_1} - s_{\zeta_1} s_{\beta_1} s_{\theta_1} & c_{\beta_1} s_{\theta_1} \\ -c_{\zeta_1} s_{\beta_1} c_{\theta_1} + s_{\theta_1} s_{\zeta_1} & -c_{\zeta_1} s_{\theta_1} - s_{\zeta_1} s_{\beta_1} c_{\theta_1} & c_{\beta_1} c_{\theta_1} \end{bmatrix} \tag{9}$$

where $c_{(\ )} = \cos(\ )$, $s_{(\ )} = \sin(\ )$, and $[T]^{-1} = [T]^T$. The angular velocity is taken from figure 3:

$$\boldsymbol{\omega} = \dot{\zeta}_1 \mathbf{K} - \dot{\beta}_1 (c_{\zeta_1} \mathbf{J} - s_{\zeta_1} \mathbf{I}) + \dot{\theta}_1 \mathbf{i}$$

$$= \omega_i \mathbf{i} + \omega_j \mathbf{j} + \omega_k \mathbf{k} \tag{10}$$

where $(\dot{\ }) = \partial/\partial t(\ )$. The components of $\boldsymbol{\omega}$ along $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, determined from equations (1), (9), and (10), are

$$\left.\begin{aligned} \omega_i &= \dot{\theta}_1 + \dot{\zeta}_1 s_{\beta_1} \\ \omega_j &= -\dot{\beta}_1 c_{\theta_1} + \dot{\zeta}_1 c_{\beta_1} s_{\theta_1} \\ \omega_k &= \dot{\zeta}_1 c_{\beta_1} c_{\theta_1} + \dot{\beta}_1 s_{\theta_1} \end{aligned}\right\} \tag{11}$$

The virtual rotation vector $\boldsymbol{\delta\psi}$ can be expressed by replacement of $(\dot{\ })$ with $\delta(\ )$ in equations (10) and (11) (ref. 10). Thus

$$\boldsymbol{\delta\psi} = \delta\zeta_1 \mathbf{K} - \delta\beta_1 (c_{\zeta_1} \mathbf{J} - s_{\zeta_1} \mathbf{I}) + \delta\theta_1 \mathbf{i}$$

$$= \delta\psi_i \mathbf{i} + \delta\psi_j \mathbf{j} + \delta\psi_k \mathbf{k} \tag{12}$$

where

$$\left.\begin{aligned} \delta\psi_i &= \delta\theta_1 + s_{\beta_1} \delta\zeta_1 \\ \delta\psi_j &= -c_{\theta_1} \delta\beta_1 + c_{\beta_1} s_{\theta_1} \delta\zeta_1 \\ \delta\psi_k &= c_{\beta_1} c_{\theta_1} \delta\zeta_1 + s_{\theta_1} \delta\beta_1 \end{aligned}\right\} \tag{13}$$

The virtual rotations are needed to express the virtual work of applied external moments (such as the aerodynamic pitching moment). The torsion (or angle of twist per unit length) and the bending curvatures may be deduced with the use of Kirchhoff's kinetic analog (ref. 1) by replacing $(\cdot)$ with $(\ )^+$ in equation (10)

$$\boldsymbol{\kappa} = \zeta_1^+ \mathbf{K} - \beta_1^+ (c_{\zeta_1} \mathbf{J} - s_{\zeta_1} \mathbf{I}) + \theta_1^+ \mathbf{i}$$

$$= \kappa_i \mathbf{i} + \kappa_j \mathbf{j} + \kappa_k \mathbf{k} \tag{14}$$

where κ_i, the torsion, is

$$\kappa_i = \theta_1^+ + \zeta_1^+ s_{\beta_1} \tag{15}$$

and the components of bending curvature in the $\mathbf{j}$ and $\mathbf{k}$ directions are

$$\left.\begin{aligned} \kappa_j &= -\beta_1^+ c_{\theta_1} + \zeta_1^+ c_{\beta_1} s_{\theta_1} \\ \kappa_k &= \zeta_1^+ c_{\beta_1} c_{\theta_1} + \beta_1^+ s_{\theta_1} \end{aligned}\right\} \tag{16}$$

We now define an angle of elastic twist ϕ so that

$$(\theta_t + \phi)^+ = \kappa_i \tag{17}$$

where $\theta_t^+ = \theta_t' x^+$. We note that θ_t and ϕ can be considered together because each occurs about the beam elastic axis. This angle of elastic twist is discussed in references 12-15 and used in references 3-5 as the torsion variable. From equations (5) and (9) we find that

$$s_{\beta_1} = w^+ \tag{18}$$

Then, we differentiate equation (5b) and make use of equations (5), (7), (9), and (18) yielding

$$\zeta_1^+ = \frac{v^{++}}{\sqrt{1 - v^{+2} - w^{+2}}} + \frac{v^+ w^+ w^{++}}{(1 - w^{+2})\sqrt{1 - v^{+2} - w^{+2}}} \tag{19}$$

Substitution of equations (18) and (19) into equations (15) and (17) yields an expression for θ_1^+ in terms of variables v, w, and ϕ which does not depend on any of the angles ζ_1, β_1, θ_1:

$$\theta_1^+ = (\theta_t + \phi)^+ - \frac{w^+}{\sqrt{1 - v^{+2} - w^{+2}}} \left(v^{++} + \frac{v^+ w^+ w^{++}}{1 - w^{+2}} \right) \tag{20}$$

or

$$\theta_1 = \theta_t + \phi - \int_0^r \frac{w^+}{\sqrt{1 - v^{+2} - w^{+2}}} \left(v^{++} + \frac{v^+ w^+ w^{++}}{1 - w^{+2}} \right) dr \tag{21}$$

if $\theta_1(0) = \theta_t(0) = \theta_0$ and $\phi(0) = 0$.

The [T] matrix may now be expressed in terms of v, w, and ϕ with no dependence on ζ_1, β_1, and θ_1. We make use of equations (5), (7), and (9) obtaining

$$[T] = \begin{bmatrix} \sqrt{1 - v^{+2} - w^{+2}} & v^+ & w^+ \\ \dfrac{-v^+ c_{\theta_1} - w^+ s_{\theta_1} \sqrt{1 - v^{+2} - w^{+2}}}{\sqrt{1 - w^{+2}}} & c_{\theta_1} \dfrac{\sqrt{1 - v^{+2} - w^{+2}} - v^+ w^+ s_{\theta_1}}{\sqrt{1 - w^{+2}}} & s_{\theta_1} \sqrt{1 - w^{+2}} \\ \dfrac{v^+ s_{\theta_1} - w^+ c_{\theta_1} \sqrt{1 - v^{+2} - w^{+2}}}{\sqrt{1 - w^{+2}}} & -s_{\theta_1} \dfrac{\sqrt{1 - v^{+2} - w^{+2}} - v^+ w^+ c_{\theta_1}}{\sqrt{1 - w^{+2}}} & c_{\theta_1} \sqrt{1 - w^{+2}} \end{bmatrix} \tag{22}$$

where θ_1 is given by equation (21). This result for [T] was first obtained in reference 3 in a different manner. The components of the angular velocity vector, the virtual rotation vector, and the rotation per-unit-length vector can now be expressed in terms of v, w, and ϕ. In doing this we make use of equations (5), (7), (9), (11), (13), (15), (16), and (17):

$$\left.\begin{aligned} \omega_i &= \dot{\theta}_1 + \frac{w^+}{\sqrt{1 - v^{+2} - w^{+2}}} \left(\dot{v}^+ + \frac{v^+ w^+ \dot{w}^+}{1 - w^{+2}} \right) \\ \omega_j &= \frac{-\dot{w}^+ c_{\theta_1}}{\sqrt{1 - w^{+2}}} + \left(\dot{v}^+ + \frac{v^+ w^+ \dot{w}^+}{1 - w^{+2}} \right) \frac{s_{\theta_1} \sqrt{1 - w^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}} \\ \omega_k &= \frac{\dot{w}^+ s_{\theta_1}}{\sqrt{1 - w^{+2}}} + \left(\dot{v}^+ + \frac{v^+ w^+ \dot{w}^+}{1 - w^{+2}} \right) \frac{c_{\theta_1} \sqrt{1 - w^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}} \end{aligned}\right\} \tag{23}$$

$$\left.\begin{aligned}
\delta\psi_i &= \delta\theta_1 + \frac{w^+}{\sqrt{1 - v^{+2} - w^{+2}}}\left(\delta v^+ + \frac{v^+w^+\delta w^+}{1 - w^{+2}}\right) \\
\delta\psi_j &= \frac{-c_{\theta_1}\delta w^+}{\sqrt{1 - w^{+2}}} + \left(\delta v^+ + \frac{v^+w^+\delta w^+}{1 - w^{+2}}\right)\frac{s_{\theta_1}\sqrt{1 - w^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}} \\
\delta\psi_k &= \frac{s_{\theta_1}\delta w^+}{\sqrt{1 - w^{+2}}} + \left(\delta v^+ + \frac{v^+w^+\delta w^+}{1 - w^{+2}}\right)\frac{c_{\theta_1}\sqrt{1 - w^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}}
\end{aligned}\right\} \tag{24}$$

$$\left.\begin{aligned}
\kappa_i &= (\theta_t + \phi)^+ \\
\kappa_j &= \frac{-w^{++}c_{\theta_1}}{\sqrt{1 - w^{+2}}} + \left(v^{++} + \frac{v^+w^+w^{++}}{1 - w^{+2}}\right)\frac{s_{\theta_1}\sqrt{1 - w^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}} \\
\kappa_k &= \frac{w^{++}s_{\theta_1}}{\sqrt{1 - w^{+2}}} + \left(v^{++} + \frac{v^+w^+w^{++}}{1 - w^{+2}}\right)\frac{c_{\theta_1}\sqrt{1 - w^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}}
\end{aligned}\right\} \tag{25}$$

The variables $\dot{\theta}_1$ and $\delta\theta_1$ are obtained by differentiation and variation of θ_1 in equation (21). Equations (21) through (25) provide a complete and exact description of the deformed-beam geometry. These equations are independent of ζ_1, β_1, and θ_1 since θ_1 is expressed in terms of v, w, and ϕ.

To illustrate the uniqueness of the geometry we present the development again with a different sequence of rotations β_2, ζ_2, θ_2. We make use of equations (1) and (5), which do not depend on the sequence of rotations. The [T] matrix can be determined from the successive transformation relating x_3, y_3, z_3 to x, y, z using figure 4.

$$[T] = \begin{bmatrix}
c_{\zeta_2}c_{\beta_2} & s_{\zeta_2} & c_{\zeta_2}s_{\beta_2} \\
-c_{\beta_2}c_{\theta_2}s_{\zeta_2} - s_{\beta_2}s_{\theta_2} & c_{\theta_2}c_{\zeta_2} & s_{\theta_2}c_{\beta_2} - c_{\theta_2}s_{\zeta_2}s_{\beta_2} \\
-s_{\beta_2}c_{\theta_2} + s_{\theta_2}s_{\zeta_2}c_{\beta_2} & -s_{\theta_2}c_{\zeta_2} & c_{\beta_2}c_{\theta_2} + s_{\beta_2}s_{\theta_2}s_{\zeta_2}
\end{bmatrix} \tag{26}$$

The angular velocity vector is, from figure 4

$$\boldsymbol{\omega} = -\dot{\beta}_2\mathbf{J} + \dot{\zeta}_2(c_{\beta_2}\mathbf{K} - s_{\beta_2}\mathbf{I}) + \dot{\theta}_2\mathbf{i} \tag{27}$$

the components of which are

$$\left.\begin{aligned} \omega_i &= \dot{\theta}_2 - \dot{\beta}_2 s_{\zeta_2} \\ \omega_j &= -\dot{\beta}_2 c_{\theta_2} c_{\zeta_2} + \dot{\zeta}_2 s_{\theta_2} \\ \omega_k &= \dot{\zeta}_2 c_{\theta_2} + \dot{\beta}_2 s_{\theta_2} c_{\zeta_2} \end{aligned}\right\} \tag{28}$$

The virtual rotation components are

$$\left.\begin{aligned} \delta\psi_i &= \delta\theta_2 - s_{\zeta_2}\delta\beta_2 \\ \delta\psi_j &= -c_{\theta_2} c_{\zeta_2}\delta\beta_2 + s_{\theta_2}\delta\zeta_2 \\ \delta\psi_k &= c_{\theta_2}\delta\zeta_2 + s_{\theta_2} c_{\zeta_2}\delta\beta_2 \end{aligned}\right\} \tag{29}$$

The torsion is

$$\kappa_i = \theta_2^+ - \beta_2^+ s_{\zeta_2} \tag{30}$$

and the bending curvatures are

$$\left.\begin{aligned} \kappa_j &= -\beta_2^+ c_{\theta_2} c_{\zeta_2} + \zeta_2^+ s_{\theta_2} \\ \kappa_k &= \zeta_2^+ c_{\theta_2} + \beta_2^+ s_{\theta_2} c_{\zeta_2} \end{aligned}\right\} \tag{31}$$

Substitution of equation (17) into equation (30) yields an expression for θ_2^+

$$\theta_2^+ = (\theta_t + \phi)^+ + \beta_2^+ s_{\zeta_2} \tag{32}$$

Equations (5) and (26) lead to

$$\left.\begin{aligned} s_{\zeta_2} &= v^+ \\ \beta_2^+ &= \frac{w^{++}}{\sqrt{1 - v^{+2} - w^{+2}}} + \frac{v^+ w^+ v^{++}}{(1 - v^{+2})\sqrt{1 - v^{+2} - w^{+2}}} \end{aligned}\right\} \tag{33}$$

Substitution of equations (33) into equation (32) yields a relationship between θ_2 and v, w, and ϕ

$$\theta_2^+ = (\theta_t + \phi)^+ + \frac{v^+}{\sqrt{1 - v^{+2} - w^{+2}}}\left(w^{++} + \frac{v^+ w^+ v^{++}}{1 - v^{+2}}\right) \tag{34}$$

or

$$\theta_2 = \theta_t + \phi + \int_0^r \frac{v^+}{\sqrt{1 - v^{+2} - w^{+2}}}\left(w^{++} + \frac{v^+ w^+ v^{++}}{1 - v^{+2}}\right) dr \tag{35}$$

where $\theta_2(0) = \theta_t(0) = \theta_0$ and $\phi(0) = 0$. Equations (5), (26), and (28) through (31) can be used to express the exact deformed-beam geometry in terms of v, w, and ϕ. The [T] matrix is

$$[T] = \begin{bmatrix} \sqrt{1 - v^{+2} - w^{+2}} & v^+ & w^+ \\ \dfrac{-v^+ c_{\theta_2}\sqrt{1 - v^{+2} - w^{+2}} - w^+ s_{\theta_2}}{\sqrt{1 - v^{+2}}} & c_{\theta_2}\sqrt{1 - v^{+2}} & \dfrac{s_{\theta_2}\sqrt{1 - v^{+2} - w^{+2}} - v^+ w^+ c_{\theta_2}}{\sqrt{1 - v^{+2}}} \\ \dfrac{v^+ s_{\theta_2}\sqrt{1 - v^{+2} - w^{+2}} - w^+ c_{\theta_2}}{\sqrt{1 - v^{+2}}} & -s_{\theta_2}\sqrt{1 - v^{+2}} & \dfrac{c_{\theta_2}\sqrt{1 - v^{+2} - w^{+2}} + v^+ w^+ s_{\theta_2}}{\sqrt{1 - v^{+2}}} \end{bmatrix} \tag{36}$$

The components of the angular velocity vector, the virtual rotation vector, and the rotation per-unit-length vector can now be expressed in terms of v, w, and ϕ. In doing this we make use of equations (5), (7), (19), (26), and (28) through (31):

$$\left.\begin{aligned} \omega_i &= \dot{\theta}_2 - \frac{v^+}{\sqrt{1 - v^{+2} - w^{+2}}}\left(\dot{w}^+ + \frac{v^+ w^+ \dot{v}^+}{1 - v^{+2}}\right) \\ \omega_j &= -\left(\dot{w}^+ + \frac{v^+ w^+ \dot{v}^+}{1 - v^{+2}}\right)\frac{c_{\theta_2}\sqrt{1 - v^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}} + \frac{\dot{v}^+ s_{\theta_2}}{\sqrt{1 - v^{+2}}} \\ \omega_k &= \left(\dot{w}^+ + \frac{v^+ w^+ \dot{v}^+}{1 - v^{+2}}\right)\frac{s_{\theta_2}\sqrt{1 - v^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}} + \frac{\dot{v}^+ c_{\theta_2}}{\sqrt{1 - v^{+2}}} \end{aligned}\right\} \tag{37}$$

$$\left.\begin{aligned}
\delta\psi_i &= \delta\theta_2 - \frac{v^+}{\sqrt{1 - v^{+2} - w^{+2}}}\left(\delta w^+ + \frac{v^+ w^+ \delta v^+}{1 - v^{+2}}\right) \\
\delta\psi_j &= -\left(\delta w^+ + \frac{v^+ w^+ \delta v^+}{1 - v^{+2}}\right)\frac{c_{\theta_2}\sqrt{1 - v^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}} + \frac{\delta v^+ s_{\theta_2}}{\sqrt{1 - v^{+2}}} \\
\delta\psi_k &= \left(\delta w^+ + \frac{v^+ w^+ \delta v^+}{1 - v^{+2}}\right)\frac{s_{\theta_2}\sqrt{1 - v^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}} + \frac{\delta v^+ c_{\theta_2}}{\sqrt{1 - v^{+2}}}
\end{aligned}\right\} \tag{38}$$

$$\left.\begin{aligned}
\kappa_i &= (\theta_t + \phi)^+ \\
\kappa_j &= -\left(w^{++} + \frac{v^+ w^+ v^{++}}{1 - v^{+2}}\right)\frac{c_{\theta_2}\sqrt{1 - v^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}} + \frac{v^{++} s_{\theta_2}}{\sqrt{1 - v^{+2}}} \\
\kappa_k &= \left(w^{++} + \frac{v^+ w^+ v^{++}}{1 - v^{+2}}\right)\frac{s_{\theta_2}\sqrt{1 - v^{+2}}}{\sqrt{1 - v^{+2} - w^{+2}}} + \frac{v^{++} c_{\theta_2}}{\sqrt{1 - v^{+2}}}
\end{aligned}\right\} \tag{39}$$

where $\dot{\theta}_2$ and $\delta\theta_2$ may be obtained directly from equation (35). Equations (36) through (39), written in terms of θ_2 defined in equation (35), appear to be different from equations (22) through (25) written in terms of θ_1 defined in equation (21). It is possible to demonstrate equivalence of the two sets of equations by simply solving each of equations (21) and (35) for ϕ and equating the expressions for ϕ. The following first integral is obtained:

$$\theta_2 = \theta_1 + \tan^{-1}\left(\frac{v^+ w^+}{\sqrt{1 - v^{+2} - w^{+2}}}\right) \tag{40}$$

so that

$$\left.\begin{aligned}
s_{\theta_2} &= \frac{s_{\theta_1}\sqrt{1 - v^{+2} - w^{+2}} + v^+ w^+ c_{\theta_1}}{\sqrt{1 - v^{+2}}\sqrt{1 - w^{+2}}} \\
c_{\theta_2} &= \frac{c_{\theta_1}\sqrt{1 - v^{+2} - w^{+2}} - v^+ w^+ s_{\theta_1}}{\sqrt{1 - v^{+2}}\sqrt{1 - w^{+2}}}
\end{aligned}\right\} \tag{41}$$

These relations were also obtained in reference 15. Equations (40) and (41) can also be derived by equating the expressions for T_{23} in equations (22) and (36). Now, by substitution of equations (41) into equations (36)

through (39), the results can be shown to reduce identically to equations (22) through (25). Thus, the sequence of rotations used in the derivation is proven to be immaterial and the two sets of equations are seen to be related by a simple change of variable. The fact is further illustrated in the appendix, in which all six possible sequences are examined and shown to be equivalent.

3. COMPARISON WITH OTHER WORK

The results of the development in section 2, except for slight differences in notation, are identical to the results in references 3-5. Several discussions of the material in references 3-5 appear in the literature and it is the purpose of this section to examine these discussions.

In references 6-8, the [T] matrix, the torsion, and the components of bending curvature are obtained for both the ζ_1, β_1, θ_1 sequence and the β_2, ζ_2, θ_2 sequence of rotations. The expressions therein are approximated to second-degree accuracy in the deflections u, v, w and θ_1 or θ_2. The authors denote θ_1 and θ_2 by the same symbol even though θ_1 and θ_2 are different angles. After recognizing this slight inconsistency in notation, however, it is quite easy to verify that the geometric quantities expressed therein are equivalent to the two formulations in the preceding section. Indeed, the development in the preceding section shows that the sequence of rotations used in describing the orientation of the local principal axes of the beam cross section is arbitrary. Thus the sequence of transformations cannot influence the outcome of a stability or response analysis. The equations appear to be different but a change of variable can always be found that will transform the equations derived based on one sequence to that of any other. The application of this fundamental idea to the rigid-beam approximation of an elastic beam will be discussed in section 6.

Let us now examine the question of whether the sequence of transformations used in an elastic-beam analysis can affect the results of the analysis as discussed in reference 6. This involves demonstrating that the two developments in references 6-8 are actually equivalent to each other and to that of references 3-5 and that the [T] matrix and kinematic quantities developed in references 3-5, despite the apparent differences with the results of references 6-8, are nonetheless correct. It is convenient for comparison with references 6-8 to approximate the geometric quantities obtained in section 2 to second order in bending and torsion deflections.

Let us consider the case with $\theta_t = 0$ and v^+, w^+, θ_1, θ_2 and ϕ of $O(\varepsilon)$ so that $\varepsilon^2 << 1$. Equation (8) shows that

$$r'^2 = (1 + u')^2 + v'^2 + w'^2 = 1 + 2e \tag{42}$$

where e is the component of longitudinal strain of the elastic axis from Green's strain tensor. In engineering beam theory, e is neglected with respect to unity. Here we assume $e = O(\varepsilon^2)$ so that $r' = 1 + O(\varepsilon^2)$ and $u' = O(\varepsilon^2)$; thus, derivatives with respect to r may be replaced by

derivatives with respect to x. For comparison we write only the T_{23} and T_{32} elements of the [T] matrix, the torsion, and the bending curvatures of the ζ_1, β_1, θ_1 formulation:

$$\left.\begin{aligned} T_{23} &= \theta_1 + O(\varepsilon^3) \\ T_{32} &= -\theta_1 - v'w' + O(\varepsilon^3) \\ \kappa_i &= \theta_1' + v''w' + O(\varepsilon^3) \\ \kappa_j &= -w'' + \theta_1 v'' + O(\varepsilon^3) \\ \kappa_k &= v'' + \theta_1 w'' + O(\varepsilon^3) \end{aligned}\right\} \tag{43}$$

where equations (21), (22), and (25) have been used. For the β_2, ζ_2, θ_2 formulation these quantities are

$$\left.\begin{aligned} T_{23} &= \theta_2 - v'w' + O(\varepsilon^3) \\ T_{32} &= -\theta_2 + O(\varepsilon^3) \\ \kappa_i &= \theta_2' - v'w'' + O(\varepsilon^3) \\ \kappa_j &= -w'' + \theta_2 v'' + O(\varepsilon^3) \\ \kappa_k &= v'' + \theta_2 w'' + O(\varepsilon^3) \end{aligned}\right\} \tag{44}$$

where equations (35), (36), and (39) have been used. These quantities are identical to the corresponding expressions obtained in references 7 and 8. Equation (40) yields, to second order, the following relationship between θ_1 and θ_2:

$$\theta_2 = \theta_1 + v'w' + O(\varepsilon^3) \tag{45}$$

which, upon substitution into equations (44), produces equations (43) identically. These expressions, which form the basis for deriving the equations of motion, are thus equivalent. Equation (45) can be thought of as a change of variable. If it is applied to the equations of motion based on the β_2, ζ_2, θ_2 sequence (flap-lag-pitch) derived in reference 8, the result will be identically the other set of equations of motion in reference 8 that are based on the ζ_1, β_1, θ_1 sequence (lag-flap-pitch).

Equations (43) and (44) both reduce to the same expressions when the elastic angle of twist is introduced, from equations (20) and (21)

$$\left.\begin{aligned} \phi' &= \theta_1' + v''w' + O(\varepsilon^3) = \theta_2' - v'w'' + O(\varepsilon^3) \\ \phi &= \theta_1 + \int_0^x v''w'dx + O(\varepsilon^3) = \theta_2 - \int_0^x v'w''dx + O(\varepsilon^3) \end{aligned}\right\} \tag{46}$$

or

$$\left.\begin{aligned} \theta_1 &= \phi - \int_0^x v''w'dx + O(\varepsilon^3) \\ \theta_2 &= \phi + \int_0^x v'w''dx + O(\varepsilon^3) \end{aligned}\right\} \tag{47}$$

Now equations (43) and (44) become

$$\left.\begin{aligned} T_{23} &= \phi - \int_0^x v''w'dx + O(\varepsilon^3) \\ T_{32} &= -\phi - \int_0^x v'w''dx + O(\varepsilon^3) \\ \kappa_i &= \phi' + O(\varepsilon^3) \\ \kappa_j &= -w'' + \phi v'' + O(\varepsilon^3) \\ \kappa_k &= v'' + \phi w'' + O(\varepsilon^3) \end{aligned}\right\} \tag{48}$$

Equations (47) relate ϕ, θ_1, and θ_2 and may also be regarded as changes of variable relating the angles θ_1 and θ_2 to the elastic twist angle ϕ. When these changes of variable are applied to the development in reference 8, the two sets of equations derived therein based on ζ_1, β_1, θ_1 and based on β_2, ζ_2, θ_2 (when specialized to hovering flight) are found to be equivalent to the set of equations found in references 3-5 (which considered only the hovering flight case). Because of the nature of ϕ, this change of variable is most easily applied to the energy and virtual work expressions, however, not directly to the partial differential equations, as shown in section 5. The outcome of the analysis does not depend on the sequence of rotations and the work of references 3-5 stands as correct.

Other authors (refs. 16 and 17) have had difficulty reconciling their work with references 3-5. In reference 16 it is stated that the results of references 3-5 are in error because of an incorrect expression for torsion, a comment similar to the conclusions of references 6-8. The [T] matrix in references 3-5 is stated to be incorrect in references 16 and 17 because the pretwist angle is grouped with the torsion variable rather than being imposed before the elastic deformation. Furthermore, it is stated in reference 16

that the quantity θ_t' must appear in [T]. The pretwist θ_t can be grouped with θ_1, θ_2, and ϕ in the present formulation because all of these rotations occur about the same axis.[2] The development in section 2 of this report is exact for an Euler-Bernoulli beam, and θ_t' terms do not (and cannot, because of dimensional reasons) appear in [T]. In contrast to references 16 and 17, there is no difference in the final form of the [T] matrix that results from interchanging the order of pretwist and elastic deformations (see eq. (17)). In both references 16 and 17, the two sets of axes that the [T] matrix relates are not the same two sets of axes that the [T] matrix in this report (and in refs. 3-5) relates. In references 16 and 17, the [T] matrix relates the local beam cross-section principal axes after deformation to the local beam cross-section principal axes before deformation and not to the x, y, z axes used here. Thus, for $\theta_t \neq 0$ a direct comparison of the [T] matrix and other geometric quantities in references 16 and 17 with those of references 3-5 and the present report is not possible.

In reference 12, two of the present authors established that the lateral buckling load for uniform, slender cantilever beams was identical for two sets of equations — one in terms of θ_1 and the other in terms of ϕ. These equations were different in appearance, yet they were equivalent and yielded the identical buckling load. The development in reference 12 in terms of ϕ proceeds from the Kirchhoff-Love equations (ref. 1) which in no way depend on the sequence of rotations used to describe the cross-section orientation. Careful examination of the different approaches noted in reference 12 illustrates that it is important to understand that the two geometric angles, denoted by θ_1 and θ_2 in this report, and the elastic angle of twist ϕ are, in fact, different quantities. Some of the confusion about the influence of different sequences of rotations evident in references 6-8 apparently stems from their use of the same symbol for the three different quantities θ_1, θ_2, and ϕ.

In reference 18, nonlinear equations of motion for an Euler-Bernoulli beam are derived for the purpose of investigating stability of the beam with applied loading. Classical Euler angles are used to describe the cross-section orientation. The three axes about which rotations occur for infinitesimal angles are not mutually orthogonal for classical Euler angles, for which two rotations occur about the axis that lies along what is nominally the beam elastic axis. Thus, the torsion variable τ is defined in reference 18 as the sum of the first and third Euler angles. The torsion κ_i and the $\mathbf{I}$ component of the virtual rotation vector $\delta\psi_I = \boldsymbol{\delta\psi} \cdot \mathbf{I}$ are given to second order in reference 18 as

$$\left.\begin{aligned} \kappa_i &= \tau' + \frac{1}{2}\left(v''w' - v'w''\right) + O(\varepsilon^3) \\ \delta\psi_I &= \delta\tau - \frac{1}{2}\left(\delta v'w' - v'\delta w'\right) + O(\varepsilon^3) \end{aligned}\right\} \tag{49}$$

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[2]This is <u>not</u> true of the other four sequences treated in the appendix, however. There, angular displacements θ_3, θ_4, θ_5 and θ_6 do not rotate the beam cross section about the deformed-beam elastic axis.

These quantities, in terms of θ_1 and θ_2, are, from equations (12), (25), and (40)

$$\left.\begin{aligned} \kappa_i &= \theta_1' + v''w' + O(\varepsilon^3) = \theta_2' - v'w'' + O(\varepsilon^3) \\ \delta\psi_I &= \delta\theta_1 + v'\delta w' + O(\varepsilon^3) = \delta\theta_2 - w'\delta v' + O(\varepsilon^3) \end{aligned}\right\} \tag{50}$$

Equations (49) and (50) can be shown to be equivalent to second order by use of the following changes of variable:

$$\tau = \theta_1 + \frac{v'w'}{2} + O(\varepsilon^3) = \theta_2 - \frac{v'w'}{2} + O(\varepsilon^3) \tag{51}$$

Thus, the equations of motion derived with the aid of classical Euler angles in reference 18 can be shown to be equivalent to those derived with the rotations defined in this report. To formulate the complete, exact geometry using classical Euler angles and the elastic angle of twist, ϕ is quite difficult, however, and may not even be possible because singularities appear in the integrals when deformations are small instead of large as in the present formulation. Also, the angles used in this report, unlike classical Euler angles, bear a strong resemblance to the geometric change in orientation during moderate deformations.

The results of this section show that the expressions for deformed-beam geometry developed in section 2 and in references 3-5 are equivalent to a variety of formulations found in the literature. The results of references 3-5 are also confirmed. The nature of the torsion variable ϕ and how a formulation in terms of ϕ differs from one in terms of θ_1 or θ_2 has not been treated in the literature and will be addressed below.

4. NATURE OF THE TORSION VARIABLES θ_1, θ_2, AND ϕ

In this section we will examine the nature of the torsion variables θ_1, θ_2, and ϕ. The kinematical relationships that define these variables are used to establish differences among them. An example problem is introduced to aid in illustrating these differences. The geometric relationship between θ_1 and θ_2 is examined by projecting the local cross-section principal axes on the yz plane. Finally, the elastic twist ϕ is shown to be a quasi-coordinate (ref. 10) in the space domain because it is related to physical angles through integrals, and its mathematical analogy with the axial deflection of a beam due to longitudinal strain is established.

4.1 Kinematical Differences

The elastic angle of twist ϕ is defined to be the elastic component of the integral of the torsion κ_i. Thus, from equation (17)

$$\phi = \int_0^r \kappa_i \, dr - \theta_t \tag{52}$$

The elastic torsion moment is then simply $GJ\phi^+$ (ref. 4).

The angle ϕ may be thought of as the beam twisting caused by a distributed, externally applied torsion moment acting about the local x_3 axis. The angle ϕ does not completely define the orientation of the reference axes at point P even when ζ_1 and β_1 (or v^+ and w^+) are known at P. In addition, θ_1 (or θ_2) must be calculated to obtain the complete description of the geometry of a deformed beam. This requires knowledge of ζ_1 and β_1 for all x inboard of P. Both θ_1 and θ_2 occur as rotations about x_3, the deformed-beam elastic axis (i.e., the $\mathbf{i}$ unit vector). By definition, θ_1 and θ_2 are the third angles in the two different sequences ζ_1, β_1, θ_1, and β_2, ζ_2, θ_2, respectively, as described in section 2. Since the final orientation is fixed, the angles θ_1 and θ_2 must differ. The fact that the elastic twist is not, in general, equal to either of the angles θ_1 and θ_2 is simply because of the kinematic effects of combined bending of the beam in two directions, v and w. If the bending deflections and elastic twist angle are known, θ_1 and θ_2 can be written according to equations (21) and (35).

$$\left.\begin{aligned} \theta_1 &= \theta_t + \phi - \int_0^r \frac{w^+}{\sqrt{1 - v^{+2} - w^{+2}}} \left(v^{++} + \frac{v^+ w^+ w^{++}}{1 - w^{+2}} \right) dr \\ \theta_2 &= \theta_t + \phi + \int_0^r \frac{v^+}{\sqrt{1 - v^{+2} - w^{+2}}} \left(w^{++} + \frac{v^+ w^+ v^{++}}{1 - v^{+2}} \right) dr \end{aligned}\right\} \tag{53}$$

They can be approximated quite well by the simpler expressions:

$$\left.\begin{aligned} \theta_1 &= \theta_t + \phi - \int_0^x v''w' \, dx + O(\varepsilon^3) \\ \theta_2 &= \theta_t + \phi + \int_0^x v'w'' \, dx + O(\varepsilon^3) \end{aligned}\right\} \tag{54}$$

Thus, rotations about the beam elastic axis (geometrically like elastic torsion rotations) may be produced by combined bending deflections alone, even without elastic twist ϕ or pretwist θ_t.

4.2 Example Problem

The following example problem is included to further confirm and illustrate the differences between θ_1 and θ_2 as well as to demonstrate the use of the geometric relations developed in section 2. Consider a beam that is first

inclined by an angle β_o from the x-y plane and then bent into a circular arc, without torsion or elongation, in the plane of its elastic axis and major principal axis as shown in figures 5 and 6. Thus

$$\left.\begin{aligned} \kappa_i &= 0 \\ \kappa_j &= 0 \\ \kappa_k &= -\kappa_0 = \text{constant} \end{aligned}\right\} \tag{55}$$

The equations for the space curve of the beam elastic axis are, from the geometry of figures 5 and 6,

$$u = \frac{\cos\beta_0 \sin(\kappa_0 r)}{\kappa_0} - r \; ; \quad v = \frac{1 - \cos(\kappa_0 r)}{-\kappa_0} \; ; \quad w = \frac{\sin\beta_0 \sin(\kappa_0 r)}{\kappa_0} \tag{56}$$

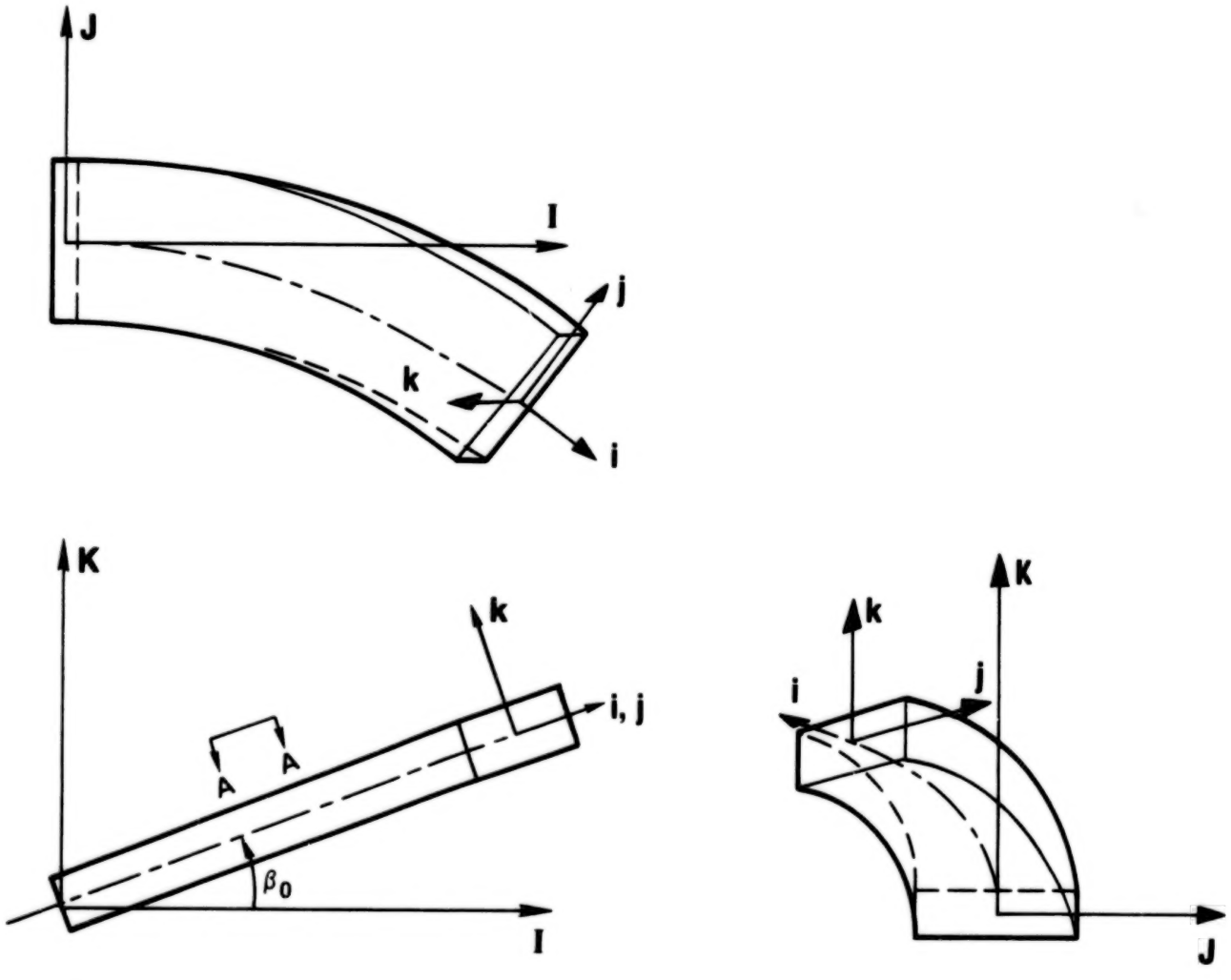

Figure 5.- Deformed beam for example problem.

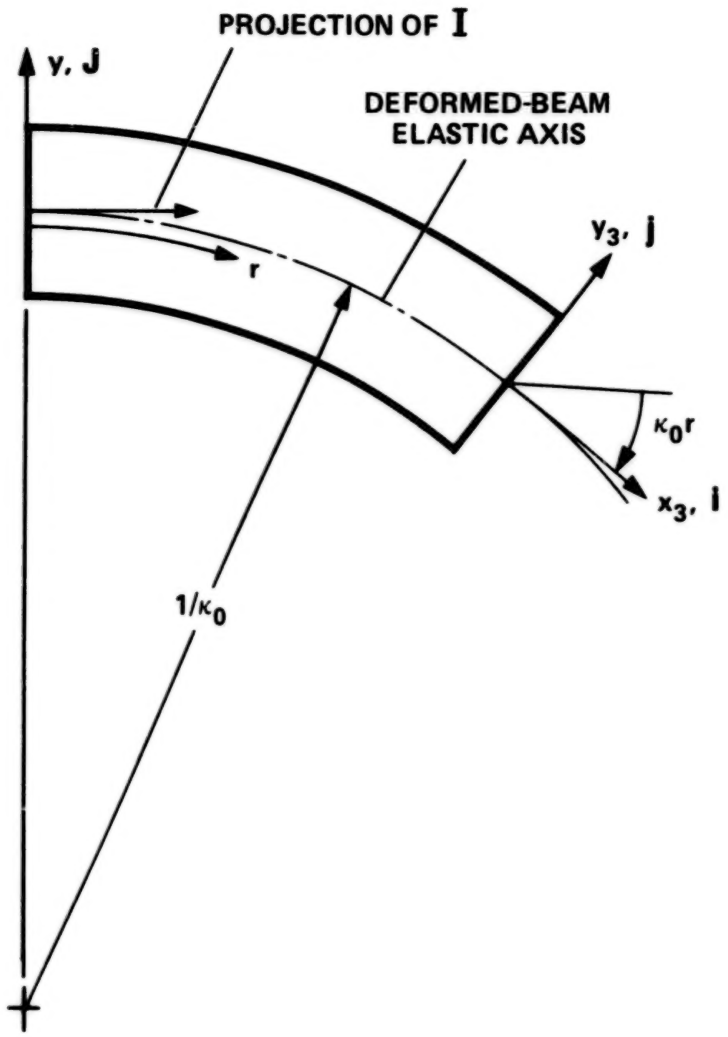

Figure 6.- View of deformed-beam elastic axis in the plane determined by **i** and **j** (view AA from fig. 5).

The matrix of direction cosines at any location is quite easily obtained from figures 5 and 6

$$\begin{Bmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{Bmatrix} = \begin{bmatrix} \cos(\kappa_0 r)\cos\beta_0 & -\sin(\kappa_0 r) & \cos(\kappa_0 r)\sin\beta_0 \\ \sin(\kappa_0 r)\cos\beta_0 & \cos(\kappa_0 r) & \sin(\kappa_0 r)\sin\beta_0 \\ -\sin\beta_0 & 0 & \cos\beta_0 \end{bmatrix} \begin{Bmatrix} \mathbf{I} \\ \mathbf{J} \\ \mathbf{K} \end{Bmatrix} \tag{57}$$

We wish to show that given u, v, w and ϕ for a beam, the equations of section 2 will provide an exact description for the deformed-beam geometry. Since κ_i is zero then $\phi = 0$. The geometric angle θ_1 can be calculated directly from substitution of equations (56) into equation (21) yielding

$$\theta_1 = \tan^{-1}[\sin(\kappa_0 r)\tan \beta_0] \tag{58}$$

Substitution of equations (58) and (56) into equation (22) for [T] yields

$$[T] = \begin{bmatrix} \cos(\kappa_0 r)\cos \beta_0 & -\sin(\kappa_0 r) & \cos(\kappa_0 r)\sin \beta_0 \\ \sin(\kappa_0 r)\cos \beta_0 & \cos(\kappa_0 r) & \sin(\kappa_0 r)\sin \beta_0 \\ -\sin \beta_0 & 0 & \cos \beta_0 \end{bmatrix} \tag{59}$$

which is identical to the matrix of direction cosines defining the example deformed beam in equation (57). It can be verified by substitution of equations (56) and (58) into the bending curvature formulas, equations (25), that the exact relations defining the problem in equations (55) are recovered. Thus the formulas of section 2 provide means of exactly determining the orientation of the deformed-beam cross section when the elastic bending and elastic torsion deflections are specified.

The same results for the transformation matrix and bending curvatures are obtained if one uses the expressions derived from the β_2, ζ_2, θ_2 rotation sequence, equations (35), (36), and (39). Proceeding as above but starting with equation (35) we find

$$\theta_2 = 0 \tag{60}$$

and from equation (36)

$$[T] = \begin{bmatrix} \cos(\kappa_0 r)\cos \beta_0 & -\sin(\kappa_0 r) & \cos(\kappa_0 r)\sin \beta_0 \\ \sin(\kappa_0 r)\cos \beta_0 & \cos(\kappa_0 r) & \sin(\kappa_0 r)\sin \beta_0 \\ -\sin \beta_0 & 0 & \cos \beta_0 \end{bmatrix} \tag{61}$$

The transformation matrix is the same as obtained in equation (59) and thus again duplicates the example problem transformation matrix given in equation (57). The fact that $\theta_1 \neq \theta_2$ should clearly indicate that the differences between θ_1 and θ_2 cannot be ignored. If θ_1 and θ_2 were treated as the same angle it would not be possible to correctly describe the geometry of the example problem in terms of both θ_1 and θ_2.

4.3 Geometric Relationship Between θ_1 and θ_2

It is helpful to visualize θ_1 and θ_2 for the general case of coupled bending deflections. This may be accomplished by considering the local

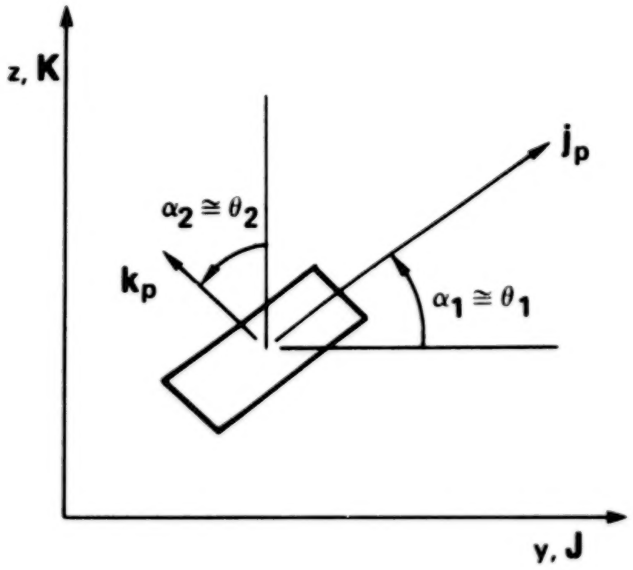

Figure 7.- Projection of the local cross-section principal axes on the y-z plane.

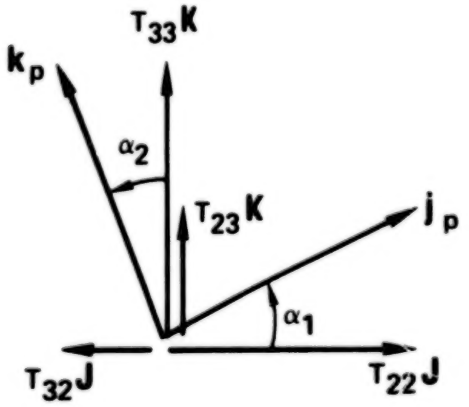

Figure 8.- The projection of unit vectors **j** and **k** on the y-z plane.

cross-section principal axes projected on the y-z plane. In figure 7 the projection of the y_3 and z_3 axes (the unit vectors **j** and **k**) are shown in the y-z plane (the plane containing unit vectors **J** and **K**). The beam cross section appears skewed because it is not in the plane of the **J** and **K** unit vectors. The projections of **j** and **k**, denoted by $\mathbf{j}_p$ and $\mathbf{k}_p$, are given by

$$\left.\begin{aligned} \mathbf{j}_p &= T_{22}\mathbf{J} + T_{23}\mathbf{K} \\ \mathbf{k}_p &= T_{32}\mathbf{J} + T_{33}\mathbf{K} \end{aligned}\right\} \tag{62}$$

By inspection from figure 8

$$\left.\begin{aligned} \sin \alpha_1 &= \frac{T_{23}}{\sqrt{T_{22}^2 + T_{23}^2}} \\ \sin \alpha_2 &= \frac{-T_{32}}{\sqrt{T_{32}^2 + T_{33}^2}} \end{aligned}\right\} \tag{63}$$

Expressions for the matrix components T_{23} and T_{32} in terms of θ_1 or θ_2 can be found in equations (22) or (36), respectively

$$\left.\begin{aligned} T_{23} &= s_{\theta_1}\sqrt{1 - w^{+2}} \\ T_{32} &= -s_{\theta_2}\sqrt{1 - v^{+2}} \end{aligned}\right\} \tag{64}$$

Combining equations (63) and (64) and dropping higher-order terms, we find that

$$\left.\begin{aligned} \alpha_1 &= \theta_1 + O(\varepsilon^3) \\ \alpha_2 &= \theta_2 + O(\varepsilon^3) \end{aligned}\right\} \tag{65}$$

and, according to equation (45)

$$\theta_2 - \theta_1 = v'w' + O(\varepsilon^3) \tag{66}$$

These results show that for small angles, θ_1 and θ_2 are essentially the kinematic rotations of the **j** and **k** axes, respectively, about the **i** axis. The fact that they are not, in general, equal is because the **j-k** plane does not remain parallel to the **J-K** plane.

Returning to figure 5, the view of the beam in the direction of the **I** axis provides a clear illustration of the kinematic rotation of the beam. The orientation of the projections of the **j** and **k** axes in figure 5 is consistent with the geometric interpretation of θ_1 and θ_2 in equations (65) and (66) and with the example calculations showing that $\theta_1 \cong -v'w'$ and $\theta_2 = 0$. These kinematic rotations, devoid of elastic twist, were previously identified in reference 3 where their effect on the blade pitch angle was noted. Hence, the term "kinematic pitch rotation" was used to describe these effects.

4.4 Mathematical Analogy of the Quasi-Coordinates ϕ and u_e

It is well known that even in the absence of longitudinal strain of the elastic axis, there may be significant nonzero axial deflection of a beam in bending. The above calculations show that even in the absence of torsional (shear) strain, there may be some nonzero pitch rotation of the cross section of a bent beam. To examine this analogy in detail we must first develop the appropriate relations concerning axial deflection. The longitudinal strain of the elastic axis may be explicitly determined from the relationship of x and r given in equation (42). Thus

$$e = u' + \frac{u'^2}{2} + \frac{v'^2}{2} + \frac{w'^2}{2} \tag{67}$$

We now let $u = u_e + u_b$ where u_b is a kinematic axial deflection due to bending only and u_e is the axial deflection due to longitudinal strain of the elastic axis. The strain becomes

$$e = u_e' + u_b' + \frac{(u_e' + u_b')^2}{2} + \frac{v'^2}{2} + \frac{w'^2}{2} \tag{68}$$

The strain must vanish when $u_e' = 0$. Thus

$$u_b' + \frac{u_b'^2}{2} + \frac{v'^2}{2} + \frac{w'^2}{2} = 0 \tag{69}$$

or

$$u_b' = \sqrt{1 - v'^2 - w'^2} - 1 \tag{70}$$

The kinematic axial deflection, sometimes called foreshortening (refs. 19-21), is then given by

$$u_b = \int_0^x \sqrt{1 - v'^2 - w'^2}\, dx - x$$

$$= -\frac{1}{2}(v'^2 + w'^2) + O(\varepsilon^4) \tag{71}$$

Thus

$$u_e = u + x - \int_0^x \sqrt{1 - v'^2 - w'^2}\, dx$$

$$= u + \frac{1}{2}\int_0^x (v'^2 + w'^2)dx + O(\varepsilon^4) \tag{72}$$

Comparison of equations (72) and (46) reveals that u_e and ϕ are mathematical analogs. The deflection u_e is devoid of kinematic shortening due to bending. The elastic twist ϕ is devoid of kinematic pitch rotation due to bending. In the time domain, quasi-coordinates (ref. 10) are usually based on integrals of either components of velocity or angular velocity. By Kirchhoff's kinetic analog (ref. 1) the angular velocity components carry over to bending curvature and torsion relations while the velocity component along the elastic axis can be shown to correspond to a longitudinal strain approximately equal to e. To obtain the latter correspondence, the velocity of point P must be written subject to the constraint that the velocity vector is tangent to the beam elastic axis. The component of strain from the analog is the difference of the velocities, for $(\dot{}) = (\ )^+$, with and without elongation of the elastic axis. Hence, both u_e and ϕ are denoted as quasi-coordinates in the space domain because they are related to the physical displacement u and angles θ_1 and θ_2 through integrals and because their derivatives have velocity and angular velocity components as kinetic analogs.

5. QUASI-COORDINATES IN THE SPACE DOMAIN AND HAMILTON'S PRINCIPLE

The angle of elastic twist ϕ has been shown in section 4 to be a quasi-coordinate in the space domain similar to the elastic component of axial deflection u_e. Although several derivations appear in the literature, making use of at least one of these variables (refs. 4-6, 8 and 19-21), it is well known that the standard form of Lagrange's equations (and thus Hamilton's principle) is not adequate when quasi-coordinates in the time domain are used (e.g., see ref. 10). It is natural to ask if there are similar obstacles when the above quasi-coordinates are used in the standard form of Hamilton's principle. To answer this question we will consider both a general development and a simple example derivation.

5.1 General Development

We will first write symbolically the standard extended form of Hamilton's principle for nonconservative systems using u, v, w, and θ_1. For simplicity, we will consider only a linearized stability analysis so that time dependence is factored out of every term. We then will rigorously take the variation. The change of variable will be made to variables u_e, v, w, and ϕ after the variation. The resulting expression, if there are no obstacles, should correspond identically to the expression obtained by starting the formulation in terms of u_e, v, w, and ϕ and then taking the variation.

The standard extended form of Hamilton's principle for an unpretwisted, nonwarping Euler-Bernoulli beam with variables u, v, w, and θ_1 is given by

$$\delta \int_{t_1}^{t_2} \int_0^L H(u, u', \dot{u}, v, v', v'', \dot{v}, w, w', w'', \dot{w}, \theta_1, \theta_1', \dot{\theta}_1)dx\, dt + \int_{t_1}^{t_2} \int_0^L \underline{\delta W}\, dx\, dt = 0 \qquad (73)$$

where H is the kinetic energy minus the potential energy and $\underline{\delta W}$ is the virtual work of the nonconservative aerodynamic forces. To simplify the following development, equation (73) is written for a linearized stability analysis, with time dependence factored out of every term or a static analysis without time dependence. The variation of H when expanded yields

$$\int_0^L \left(\frac{\partial H}{\partial u}\delta u + \frac{\partial H}{\partial u'}\delta u' + \frac{\partial H}{\partial v}\delta v + \frac{\partial H}{\partial v'}\delta v' + \frac{\partial H}{\partial v''}\delta v'' + \frac{\partial H}{\partial w}\delta w + \frac{\partial H}{\partial w'}\delta w' + \frac{\partial H}{\partial w''}\delta w'' + \frac{\partial H}{\partial \theta_1}\delta\theta_1 + \frac{\partial H}{\partial \theta_1'}\delta\theta_1' + L_x\delta u + L_y\delta v + L_z\delta w + M_{x_3}\delta\psi_i \right) dx = 0 \qquad (74)$$

where the virtual work has been expressed in terms of the generalized applied loads and $\delta\psi_i = \delta\theta_1 + w'\delta v'$ to second order. The forces are resolved along the x, y, z axes, respectively, and the pitching moment acts about the x_3 axis. We now make the change of variable that transforms equation (74) to a different expression which involves u_e and ϕ. The relationships between u, θ_1, and u_e, ϕ are, to second order, from equations (46) and (72)

$$\left.\begin{aligned} u &= u_e - \frac{1}{2}\int_0^x (v'^2 + w'^2)dx \\ \theta_1 &= \phi - \int_0^x v''w'dx \end{aligned}\right\} \qquad (75)$$

Differentiation of equations (75) yields

$$\left.\begin{aligned} u' &= u'_e - \frac{v'^2}{2} - \frac{w'^2}{2} \\ \theta'_1 &= \phi' - v''w' \end{aligned}\right\} \tag{76}$$

We now consider $H = \bar{H}(u_e, u'_e, v, v', v'', w, w', w'', \phi, \phi')$ and note that

$$\left.\begin{aligned}
&\frac{\partial H}{\partial u}\,\delta u = \frac{\partial \bar{H}}{\partial u_e}\frac{\partial u_e}{\partial u}\,\delta u = \frac{\partial \bar{H}}{\partial u_e}\,\delta u \quad ; \qquad \frac{\partial H}{\partial u'}\,\delta u' = \frac{\partial \bar{H}}{\partial u'_e}\frac{\partial u'_e}{\partial u'}\,\delta u' = \frac{\partial \bar{H}}{\partial u'_e}\,\delta u' \\
&\frac{\partial H}{\partial \theta_1}\,\delta\theta_1 = \frac{\partial \bar{H}}{\partial \phi}\frac{\partial \phi}{\partial \theta_1}\,\delta\theta_1 = \frac{\partial \bar{H}}{\partial \phi}\,\delta\theta_1 \quad ; \qquad \frac{\partial H}{\partial \theta'_1}\,\delta\theta'_1 = \frac{\partial \bar{H}}{\partial \phi'}\frac{\partial \phi'}{\partial \theta'_1}\,\delta\theta'_1 = \frac{\partial \bar{H}}{\partial \phi'}\,\delta\theta'_1 \\
&\frac{\partial H}{\partial v'}\,\delta v' = \frac{\partial \bar{H}}{\partial v'}\,\delta v' + \frac{\partial \bar{H}}{\partial u_e}\frac{\partial u_e}{\partial v'}\,\delta v' + \frac{\partial \bar{H}}{\partial u'_e}\frac{\partial u'_e}{\partial v'}\,\delta v' \\
&\qquad = \frac{\partial \bar{H}}{\partial v'}\,\delta v' + \frac{\partial \bar{H}}{\partial u_e}\int_0^x v'\delta v'dx + \frac{\partial \bar{H}}{\partial u'_e}\,v'\delta v' \\
&\frac{\partial H}{\partial v''}\,\delta v'' = \frac{\partial \bar{H}}{\partial v''}\,\delta v'' + \frac{\partial \bar{H}}{\partial \phi}\frac{\partial \phi}{\partial v''}\,\delta v'' + \frac{\partial \bar{H}}{\partial \phi'}\frac{\partial \phi'}{\partial v''}\,\delta v'' \\
&\qquad = \frac{\partial \bar{H}}{\partial v''}\,\delta v'' + \frac{\partial \bar{H}}{\partial \phi}\int_0^x w'\delta v''dx + \frac{\partial \bar{H}}{\partial \phi'}\,w'\delta v'' \\
&\frac{\partial H}{\partial w'}\,\delta w' = \frac{\partial \bar{H}}{\partial w'}\,\delta w' + \frac{\partial \bar{H}}{\partial u_e}\frac{\partial u_e}{\partial w'}\,\delta w' + \frac{\partial \bar{H}}{\partial u'_e}\frac{\partial u'_e}{\partial w'}\,\delta w' + \frac{\partial \bar{H}}{\partial \phi}\frac{\partial \phi}{\partial w'}\,\delta w' + \frac{\partial \bar{H}}{\partial \phi'}\frac{\partial \phi'}{\partial w'}\,\delta w' \\
&\qquad = \frac{\partial \bar{H}}{\partial w'}\,\delta w' + \frac{\partial \bar{H}}{\partial u_e}\int_0^x w'\delta w'dx + \frac{\partial \bar{H}}{\partial u'_e}\,w'\delta w' + \frac{\partial \bar{H}}{\partial \phi}\int_0^x v''\delta w'dx + \frac{\partial \bar{H}}{\partial \phi'}\,v''\delta w'
\end{aligned}\right\} \tag{77}$$

It should be noted that the above variation in terms of ϕ (or u_e) could not have been made if ϕ' (or u'_e) were also an explicit function of θ_1 (or u), as might occur for general quasi-coordinates. Substitution of equation (76) into equation (74) yields

$$\int_0^L \Bigg[\frac{\partial \bar{H}}{\partial u_e}\delta u + \frac{\partial \bar{H}}{\partial u_e'}\delta u' + \frac{\partial \bar{H}}{\partial v}\delta v + \frac{\partial \bar{H}}{\partial v'}\delta v' + \underline{\frac{\partial \bar{H}}{\partial u_e}\int_0^x v'\delta v'dx} + \underline{\frac{\partial \bar{H}}{\partial u_e'}v'\delta v'} + \frac{\partial \bar{H}}{\partial v''}\delta v''$$

$$+ \underline{\underline{\frac{\partial \bar{H}}{\partial \phi}\int_0^y w'\delta v''dx}} + \underline{\underline{\frac{\partial \bar{H}}{\partial \phi'}w'\delta v''}} + \frac{\partial \bar{H}}{\partial w}\delta w + \frac{\partial \bar{H}}{\partial w'}\delta w' + \underline{\frac{\partial \bar{H}}{\partial u_e}\int_0^x w'\delta w'dx} + \underline{\frac{\partial \bar{H}}{\partial u_e'}w'\delta w'}$$

$$+ \underline{\underline{\frac{\partial \bar{H}}{\partial \phi}\int_0^x v''\delta w'dx}} + \underline{\underline{\frac{\partial \bar{H}}{\partial \phi'}v''\delta w'}} + \frac{\partial \bar{H}}{\partial w''}\delta w'' + \frac{\partial \bar{H}}{\partial \phi}\delta\theta_1 + \frac{\partial \bar{H}}{\partial \phi'}\delta\theta_1' + L_x\delta u + L_y\delta v$$

$$+ L_z\delta w + M_{x_3}(\delta\theta_1 + w'\delta v')\Bigg]dx = 0 \qquad (78)$$

Up to now the change of variables in Hamilton's principle is equivalent to a direct substitution for u_e and ϕ into the partial differential equations corresponding to equation (74). The single underlined terms appear in the v and w equations from u_e, and the double underlined terms are from ϕ. Not all of these terms are necessarily visible at this step, however, depending on the order of terms that are retained. The example given below should clarify the nature of these terms. We now substitute for $\delta\theta_1$, $\delta\theta_1'$, δu, and $\delta u'$ in equation (78)

$$\left.\begin{aligned} \delta\theta_1 &= \delta\phi - \int_0^x (w'\delta v'' + v''\delta w')dx \\ \delta\theta_1' &= \delta\phi' - w'\delta v'' - v''\delta w' \\ \delta u &= \delta u_e - \int_0^x (v'\delta v' + w'\delta w')dx \\ \delta u' &= \delta u_e' - v'\delta v' - w'\delta w' \end{aligned}\right\} \qquad (79)$$

All of the underlined terms cancel out leaving

$$\int_0^L \Bigg\{ \frac{\partial \bar{H}}{\partial u_e}\delta u_e + \frac{\partial \bar{H}}{\partial u_e'}\delta u_e' + \frac{\partial \bar{H}}{\partial v}\delta v + \frac{\partial \bar{H}}{\partial v'}\delta v' + \frac{\partial \bar{H}}{\partial v''}\delta v'' + \frac{\partial \bar{H}}{\partial w}\delta w + \frac{\partial \bar{H}}{\partial w'}\delta w' + \frac{\partial \bar{H}}{\partial w''}\delta w''$$

$$+ \frac{\partial \bar{H}}{\partial \phi}\delta\phi + \frac{\partial \bar{H}}{\partial \phi'}\delta\phi' + L_x\left[\delta u_e - \int_0^x (v'\delta v' + w'\delta w')dx_1\right] + L_y\delta v + L_z\delta w$$

$$+ M_{x_3}\left[\delta\phi + \int_0^x (w''\delta v' - v''\delta w')dx_1\right]\Bigg\}dx = 0 \qquad (80)$$

This is precisely the same expression that would have resulted from starting with the quasi-coordinates and taking the variation directly. Since the virtual work term involves only a straightforward change of variable, Hamilton's principle is valid when written in terms of quasi-coordinates u_e and ϕ. Thus the application of Hamilton's principle with variables u_e and ϕ reduces to

$$\delta \int_0^L \bar{H}\, dx + \int_0^L (L_x \delta u_e + L_y \delta v + L_z \delta w + M_{x_3} \delta\phi) dx - \int_0^L \left[\left(v' \int_x^L L_x\, dx_1 \right. \right.$$

$$\left. - w'' \int_x^L M_{x_3}\, dx_1 \right) \delta v' + \left(w' \int_x^L L_x\, dx_1 + v'' \int_x^L M_{x_3}\, dx_1 \right) \delta w' \Bigg] dx \qquad (81)$$

where the following identity was used:

$$\int_0^L f(x) \int_0^x g(x_1) dx_1\, dx = \int_0^L g(x) \int_x^L f(x_1) dx_1\, dx \qquad (82)$$

It is observed that the use of the quasi-coordinates u_e and ϕ has the effect of replacing the terms underlined in equation (78) with integrals of the applied loads from both the u and θ_1 equations.

5.2 Example Derivation

An actual derivation of equations will be helpful in understanding the results from the general development above. Consider a rotating cantilever beam that undergoes coupled axial deflection and flap (out-of-plane) bending only. For simplicity we consider only the static equilibrium equations (no time dependence) for a beam with a vertical distributed load proportional to $x + u$ and an axial distributed load P. The beam is allowed to be extensional in that elongation of the elastic axis is allowed. The important special case where the elastic axis is inextensional is also considered. The position vector for a point at a distance ξ from the elastic axis is

$$\mathbf{r} = (x + u)\mathbf{I} + w\mathbf{K} + \xi\mathbf{k} \qquad (83)$$

and the axial strain, neglecting elongations with respect to unity (ref. 22), is

$$e_{xx} = e + \xi\kappa_j = u' + \frac{u'^2}{2} + \frac{w'^2}{2} - \frac{\xi w''}{\sqrt{1 - w'^2}}$$

$$\cong u' + \frac{w'^2}{2} - \xi w'' \left(1 + \frac{1}{2} w'^2 \right) \qquad (84)$$

Application of Hamilton's principle yields

$$\int_0^L \left\{ EA\left(u' + \frac{w'^2}{2}\right)(\delta u' + w'\delta w') + EIw''[(1 + w'^2)\delta w'' + w''w'\delta w'] \right.$$

$$\left. - P\delta u - m\Omega^2(x + u)\delta u - K(x + u)\delta w \right\} dx = o \qquad (85)$$

where K is a given constant, $A = \iint dA$ and $I = \iint \xi^2 dA$. The Euler-Lagrange equations and boundary conditions follow immediately:

$$-\left[EA\left(u' + \frac{w'^2}{2}\right)\right]' - m\Omega^2(x + u) = P \qquad (86a)$$

$$-\left[EA\left(u' + \frac{w'^2}{2}\right)w'\right]' + [EIw''(1 + w'^2)]'' - (EIw''^2w')' = K(x + u) \qquad (86b)$$

$$\left.\begin{aligned} u &= 0 \\ w &= 0 \\ w' &= 0 \end{aligned}\right\} \; x = 0 \qquad\qquad \left.\begin{aligned} EA\left(u' + \frac{w'^2}{2}\right) &= 0 \\ EIw'' &= 0 \\ (EIw'')' &= 0 \end{aligned}\right\} \; x = L \qquad (86c)$$

Let us consider for the moment the case of an inextensional beam. To specialize the equations we must do two things: (1) solve equation (86a) for $EA[u' + (w'^2/2)]$ and substitute the result in equation (86b); and then (2) solve for u in terms of w and substitute into equation (86b). Although $u' + (w'^2/2) = 0$ for inextensibility, $EA \to \infty$ for this case and the product $EA[u' + (w'^2/2)]$ is finite, and nonzero in general. For step (1)

$$EA\left(u' + \frac{w'^2}{2}\right) = \int_x^L [m\Omega^2(x_1 + u) + P]dx_1 = T \qquad (87)$$

Step (2) yields

$$u' = -\frac{w'^2}{2} + \frac{T}{EA} \qquad (88)$$

or, for $EA \to \infty$

$$u = -\frac{1}{2}\int_0^x w'^2 dx \qquad (89)$$

Thus, the single w equation for an inextensional beam becomes

$$-\left\{w'\int_x^L\left[m\Omega^2\left(x_1-\frac{1}{2}\int_0^{x_1}w'^2dx_2\right)+P\right]dx_1\right\}'+[EIw''(1+w'^2)]''-(EIw''^2w')'$$
$$=K\left(x-\frac{1}{2}\int_0^x w'^2dx_1\right) \qquad (90)$$

with boundary conditions as above.

We now return to the extensional case and make use of the quasi-coordinate u_e so that

$$\left.\begin{aligned} u &= u_e-\frac{1}{2}\int_0^x w'^2dx \\ u' &= u_e'-\frac{w'^2}{2} \\ EA\left(u'+\frac{w'^2}{2}\right) &= EAu_e' \end{aligned}\right\} \qquad (91)$$

Substitution into Hamilton's principle, equation (85), yields

$$\int_0^L\left\{EAu_e'(\delta u'+\underline{w'\delta w'})+EIw''[(1+w'^2)\delta w''+w''w'\delta w']-P\delta u\right.$$
$$\left.-m\Omega^2\left(x+u_e-\frac{1}{2}\int_0^x w'^2dx_1\right)\delta u-K\left(x+u_e-\frac{1}{2}\int_0^x w'^2dx_1\right)\delta w\right\}dx=0 \qquad (92)$$

The Euler-Lagrange equations are

$$-(EAu_e')'-m\Omega^2\left(x+u_e-\frac{1}{2}\int_0^x w'^2dx_1\right)=P \qquad (93a)$$

$$-\underline{(EAu_e'w')'}+[EIw''(1+w'^2)]''-(EIw''^2w')'=K\left(x+u_e-\frac{1}{2}\int_0^x w'^2dx_1\right) \qquad (93b)$$

with boundary conditions as above. The underlined term in equations (92) and (93b) corresponds to $(\partial\bar{H}/\partial u_e')w'\delta w'$ in the Hamiltonian (see equation (78)). Equations (93a) and (93b) are identical to the result of substituting equations (91) into equations (86a) and (86b). For the case of inextensibility, we still must follow two steps: (1) solve equation (93a) for EAu_e' and substitute into equation (93b); and (2) substitute $u_e=0$ into equation (93b). The result is equation (90).

To express Hamilton's principle in terms of u_e and w alone requires substitution of

$$\left.\begin{aligned} \delta u &= \delta u_e - \int_0^x w'\delta w' dx \\ \delta u' &= \delta u_e' - w'\delta w' \end{aligned}\right\} \tag{94}$$

yielding

$$\int_0^L \left\{ EAu_e'\delta u_e' + EIw''[(1 + w'^2)\delta w'' + w'w''\delta w'] - P + m\Omega^2\left(x + u_e - \frac{1}{2}\int_0^x w'^2 dx_1\right)\left(\delta u_e - \int_0^x w'\delta w' dx_1\right) - K\left(x + u_e - \frac{1}{2}\int_0^x w'^2 dx_1\right)\delta w \right\} dx = 0 \tag{95}$$

The Euler-Lagrange equations are

$$-(EAu_e')' - m\Omega^2\left(x + u_e - \frac{1}{2}\int_0^x w'^2 dx_1\right) = P \tag{96a}$$

$$-\left\{ w' \int_0^L \left[m\Omega^2\left(x_1 + u_e - \frac{1}{2}\int_0^{x_1} w'^2 dx_2\right) + P \right] dx_1 \right\}' + [EIw''(1 + w'^2)]'' - (EIw''^2 w')' = K\left(x + u_e - \frac{1}{2}\int_0^x w'^2 dx_1\right) \tag{96b}$$

For the case of inextensibility, we simply set $u_e = 0$ in equation (96b) and obtain equation (90) directly.

Although no EA terms appear in equation (96b), it is still suitable for use, along with equation (96a), when EA is finite. The reason for this is now clear: when the analysis is formulated in terms of the quasi-coordinate u_e at the outset, the substitution for the tension EAu_e' is automatically taken care of. The rest of equation (96a) is, in effect, integrated and substituted automatically. This example explains why no EAu_e' terms appear in the v and w equations of references 19-21. They do appear in the v and w equations of reference 8 because the case of the tension axis being offset from the elastic axis is considered.

At this point, it seems appropriate to comment on the advantages of using the quasi-coordinates u_e and ϕ. Both variables tend in some cases to simplify the derivations of actual equations of motion (refs. 3-5, 8, and 19-21). It is also evident that taking the limit for infinite axial or torsional rigidity may be somewhat simpler.

6. IN THE LIMIT OF INFINITE TORSIONAL RIGIDITY

Another way of looking at the question of the equivalence of the equations in references 3-5 and the two sets of equations in reference 8, demonstrated in section 3, is to consider the special limiting case of infinite torsional rigidity. It is clear that no matter what variables are used in the analysis for torsion there is a unique set of equations governing bending deflections v and w when GJ tends to infinity. For that case, all GJ terms must vanish from the v and w equations in order to preserve the proper mathematical structure of the equations. This is similar to the result of the last section in which all EA terms vanished for the special case of infinite EA.

For equations written in terms of ϕ, such as those of references 3-5, there are no $GJ\phi'$ terms in the v and w equations just as there are no EAu_e' terms in the v and w equations of references 19-21. With the ordering scheme of references 3-5, all of the terms from the ϕ equation that are integrated and substituted into the v and w equations automatically, through the process of using the quasi-coordinate ϕ, turn out to be negligible (i.e., as stated in reference 3, the torsion moment is one order of ε smaller than the bending moments). For infinite GJ, the angle of twist per unit length is simply the pretwist component θ_t'; the elastic twist ϕ and its variation $\delta\phi$ are simply set equal to zero because elastic twist about the elastic axis cannot occur.

For equations written in terms of θ_1 or θ_2, the infinite torsional rigidity constraint is more involved. If θ_1 or θ_2 were simply set to zero nonzero GJ coefficients would remain and the equations would break down. Instead, the GJ coefficient, equal to $\kappa_i - \theta_t^+$, must be set equal to zero and used to eliminate θ_1 or θ_2. The following kinematic relations hold to second order (see eqs. (46)):

$$\left.\begin{aligned} \lim_{GJ\to\infty} \theta_1 &= -\int_0^x v''w'dx + O(\varepsilon^3) \\ \lim_{GJ\to\infty} \theta_2 &= \int_0^x v'w''dx + O(\varepsilon^3) \end{aligned}\right\} \tag{97}$$

Then, the elastic torsion moment $GJ(\kappa_i - \theta_t^+)$ which is finite and not necessarily zero, must be determined from the torsion equation and substituted into the v and w equations. In this way all θ_1 or θ_2 terms and all GJ terms are eliminated from the v and w equations. Finally, if the aerodynamic pitching moment is not neglected, the virtual rotation in terms of v and w alone must be determined from equations (24) or (38) and (97):

$$\lim_{GJ\to\infty} \delta\psi_i = \int_0^x (w''\delta v' - v''\delta w')dx + O(\varepsilon^3) \tag{98}$$

for either θ_1 or θ_2. The total effect of pitching moment is then in the terms $-(w''\int_x^L M_{x_3} dx_1)'$ in the v equation and $(v''\int_x^L M_{x_3} dx_1)'$ in the w equation. It may be shown that these operations will yield a single set of equations in v and w regardless of which variable θ_1 or θ_2 is used. The only second-order term is the integral term correction to the pitch angle, the "kinematic pitch rotation" term, first identified in reference 3. The other places that the integrals appear are in third- and higher-order terms and not particularly significant. Therefore, it is not correct to simply set θ_1 or θ_2 equal to zero for general nonlinear analysis of torsionally rigid beams. In fact, when any physical displacement variable u, v, w, θ_1, θ_2 that is part of nonlinear Euler-Bernoulli beam equations must be eliminated because of infinite axial, bending or torsion stiffness, it must be done by a process similar to the one outlined above. We have shown that $u \neq 0$ when $EA \to \infty$, and $\theta_1 \neq \theta_2 \neq 0$ when $GJ \to \infty$. It is just as easily shown that neither v nor w vanishes when the bending stiffness in that direction tends to infinity. A single set of equations, consistent to second order, for a torsionally rigid rotating beam can be obtained by simply setting $\phi = 0$ in references 4 and 5 or by going through the process outlined above for either set of the equations in reference 8.

Another result of equations (97) is that three angles are required to describe the general orientation of a reference frame in space. Thus, even for infinite torsional rigidity it is possible for beam bending deformations to place beam elements in such a position that transformations based on only two angles cannot describe the orientation of a beam element. This is seen in the example problem of section 4, in which three nonzero angles may be required to describe the orientation of a beam element, despite the fact that $\kappa_i = 0$. The required third angles are given to second order by equations (97).

7. RIGID BEAM APPROXIMATIONS

When cantilevered elastic beams are modeled by an approximate system that substitutes a rigid, hinged, spring-restrained rod for the elastic beam, three degrees of freedom are often considered: ζ, an angular deflection in the x-y plane (when $\beta = \theta = 0$) or lead-lag; β, an angular deflection out of the x-y plane (when $\zeta = \theta = 0$) or flap; and θ, the pitch angle (when $\zeta = \beta = 0$). The nonlinear equations for this case are especially interesting because six different hinge arrangements can be defined which will lead to six different sets of nonlinear equations (the linear equations being the same). Simpler models, assuming no pitch degree of freedom, have been used in preliminary investigations of helicopter rotor blade stability (e.g., ref. 23). For this case there are two possibilities: flap-lag, where the lead-lag hinge flaps with the blade, and lag-flap, where the flap hinge lead-lags with the blade. The two hinge arrangements define two different physical systems with different dynamic properties although the differences are not large. Differences in the stability of these two systems were identified and discussed in reference 3, and reference 6 added further numerical results and extensive discussion. It was concluded in reference 6 that two different sets of

equations for a single *elastic* beam would result from different rotation sequences and that this could be expected to influence the stability of the elastic-beam model. This incorrect expectation was based on the correct observation that rigid-beam models with two different hinge arrangements exhibited differences in stability. In fact, however, a single set of equations for an elastic beam can be used to develop the exact equations for a rigid-beam model with any hinge arrangement. In so doing one must carefully consider the dependence of the v and w displacements on the hinge arrangement. Furthermore it can be shown that the sequence of rotations used to develop the equations is arbitrary for any rigid-beam hinge arrangement as well. For simplicity we will perform these operations for a second-order analysis.

Let us assume first a lag-flap hinge arrangement with the lead-lag hinge at the root and the flap hinge offset from the root at axial distance ε. The limit must be taken as $\varepsilon \to 0$ to recover the correct hinge arrangement even for the coincident hinge model used in reference 6. The following displacement functions are assumed

$$v = \zeta x \ , \quad v' = \zeta \ , \quad v'' = 0 \quad x \geq 0 \tag{99a}$$

but, so that the flap hinge lead-lags with the beam,

$$\left.\begin{aligned} w &= \lim_{\varepsilon \to 0} \begin{Bmatrix} 0 & x \leq \varepsilon \\ \beta(x - \varepsilon) & x > \varepsilon \end{Bmatrix} = \beta x \quad x \geq 0 \\ w' &= \lim_{\varepsilon \to 0} \begin{Bmatrix} 0 & x \leq \varepsilon \\ \beta & x > \varepsilon \end{Bmatrix} = \begin{Bmatrix} 0 & x = 0 \\ \beta & x > 0 \end{Bmatrix} \\ w'' &= \lim_{\varepsilon \to 0} \beta\delta(\varepsilon) = \beta\delta(0) \quad x \geq 0 \end{aligned}\right\} \tag{99b}$$

where $\delta(\varepsilon)$ and $\delta(0)$ refer to the Dirac delta function. For a flap-lag arrangement, we need to reverse the expressions in equations (99) so that

$$\left.\begin{aligned} & w = \beta x \ ; \quad w' = \beta \ ; \quad w'' = 0 \quad x \geq 0 \\ & v = \zeta x \quad x \geq 0 \\ & v' = \begin{Bmatrix} 0 & x = 0 \\ \zeta & x > 0 \end{Bmatrix} \\ & v'' = \zeta\delta(0) \quad x \geq 0 \end{aligned}\right\} \tag{100}$$

To reduce the elastic-beam equations in references 4 and 5 to the torsionally rigid case we simply set $\phi = 0$. The resulting equations would then contain the integrals in equations (97) that have different values for the two hinge arrangements defined by equations (99) and (100). Evaluation of the integrals for the lag-flap and flap-lag hinge arrangements yields

for lag-flap:

$$\left\{ \begin{array}{l} \displaystyle\int_0^x v''w'\,dx = 0 \\ \\ \displaystyle\int_0^x v'w''\,dx = \int_0^x \beta\delta(0)\zeta\,dx = \beta\zeta \end{array} \right\} \tag{101}$$

for flap-lag:

$$\left\{ \begin{array}{l} \displaystyle\int_0^x v''w'\,dx = \int_0^x \zeta\delta(0)\beta\,dx = \beta\zeta \\ \\ \displaystyle\int_0^x v'w''\,dx = 0 \end{array} \right\} \tag{102}$$

The differences between equations (101) and (102) result in two different sets of equations for the two different hinge arrangements, and these equations are, to second order, identical to the rigid-blade equations derived in reference 6.

For two degrees of freedom β and ζ and a given hinge arrangement, any sequence of rotations (e.g., ζ_1, β_1, θ_1 or β_2, ζ_2, θ_2) can be used to formulate the equations although it would certainly be more natural to use the angles associated with rotations about the hinges. The reconciliation of three angles with only two degrees of freedom is made by use of a holonomic constraint equation that relates θ to β and ζ (ref. 10, p. 55). For example, consider writing equations for a flap-lag hinge arrangement in terms of ζ_1, β_1, θ_1, instead of in terms β_2, ζ_2 (θ_2 is not needed because the axes about which β_2 and ζ_2 occur are along the hinge axes). Thus $\theta_2 = 0$ and equation (40) yields

$$\theta_2 = 0 = \theta_1 + \tan^{-1}\left(\frac{T_{12}T_{13}}{T_{11}}\right) \tag{103}$$

or

$$\theta_1 = -\tan^{-1}(\sin\beta_1 \tan\zeta_1) = -\beta_1\zeta_1 + O(\epsilon^3) \tag{104}$$

In fact, as long as the proper constraint is accounted for, any set of variables can be used to write the equations for any physical hinge arrangement. When the constraints are properly accounted for, the equations of motion for a given physical hinge arrangement are equivalent, although different in appearance, for any sequence of rotations used to describe the beam orientation. The dynamic behavior of the system is then independent of the transformation sequence used in describing the orientation of the blade. The equations and dynamic behavior do, however, depend on the physical arrangement of the hinges.

8. CONCLUSIONS

In this report the large deformation geometry for an Euler-Bernoulli beam has been developed. Some aspects of the beam kinematics discussed in references 3-8, 12, 16, and 17 have been clarified. The following points summarize the results of this study:

1. The large-deformation kinematics for an Euler-Bernoulli beam are developed, including the transformation matrix relating the local principal axes in the deformed state to space-fixed Cartesian axes, the components of angular velocity and virtual rotation vectors, the torsion, and the components of bending curvature. The values of all the geometric quantities are unique at a given instant in time, but the form of the expressions themselves may depend on the variables used in describing the orientation of the cross section during deformation.

2. The angles ζ_1, β_1, θ_1 are not the same as the angles β_2, ζ_2, θ_2, both sets of which are discussed in the text. Although the geometry is unique, the use of the different sequences of rotations may change the appearance of the geometric expressions.

3. The exact expressions for the large-deformation geometry that are derived in this report do not depend on the sequence of rotations used in defining the beam cross-section orientation. Some of these quantities were also derived previously in references 3-5.

4. The stability and dynamic behavior of an elastic beam are not dependent on the choice of angles used to describe the orientation of the local beam cross section during deformation. In fact, the two sets of equations derived in reference 8, based on two different sets of angles, are shown to be equivalent to each other and to the equations of references 4 and 5.

5. The exact relationship of the pitch rotation angles θ_1 and θ_2 and the elastic twist angle ϕ, established in reference 3, is verified along with the exact transformation matrix first derived therein. The integrals associated with these quantities are correct as originally derived in reference 3, and serve to show that there can be a kinematic pitch rotation of the beam cross section that depends on bending alone, even when there is no pretwist and no elastic twist.

6. The kinematic pitch rotation due to bending only that may occur even in torsionally rigid beams is seen to be mathematically analogous to a kinematic shortening that may occur even in axially rigid beams due to bending only. The torsion variable ϕ used in this report is the elastic angle of twist, which is devoid of kinematic pitch rotation due to bending alone. Its mathematical analog is the axial deflection due to longitudinal strain of the elastic axis u_e, which is devoid of kinematic axial deflection (foreshortening) due to bending only.

7. Both the elastic twist angle ϕ and the axial deflection due to longitudinal strain of the elastic axis u_e, because they are related to

physical deflections through integrals and have integrals of angular velocity and velocity components for kinetic analogs, belong to a certain class of variables called quasi-coordinates. They have both been used in Hamilton's principle without the rigorous justification provided in this report. Although not previously established, their use in references 4, 5, 8, and 19-21 is found to be valid.

8. Three angles are shown to be required in general to describe the local cross-section orientation for an elastic beam, even if the torsion stiffness is infinite. Only one set of equations exist for bending deflections v and w of an inextensional, torsionally rigid Euler-Bernoulli beam, regardless of the three angles used in the derivation.

9. When the proper mathematical limits for different hinge arrangements in a hinged rigid-beam model are imposed on the elastic-beam equations with rigid-body mode shapes, the correct hinged, rigid-beam equations, for each particular hinge arrangement, are obtained.

10. Any three angles can be used to describe the orientation of any hinged, rigid body as long as the constraints that relate the rotations about the hinges to the variables in the derivation are properly accounted for. Of course, it may be more natural to use the physical angles as variables. It is, therefore, not the transformation sequence that influences the dynamic behavior — it is the physical arrangement of the hinges.

11. The bending curvatures, torsion, and [T] matrix are developed in the appendix for six possible sequences of the angles ζ, β, and θ. Since values of the torsion and elements of the [T] matrix are fixed at a given instant in time, a change of variable may be found from either quantity to relate the geometry from any one sequence to that of any other.

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APPENDIX A

KINEMATICS FOR OTHER SEQUENCES

In the text, the development was restricted to only two of the six possible sequences of rotations ζ, β, θ. In the appendix we will present expressions for the [T] matrix, the bending curvatures, and the torsion for all six possible sequences of rotations: (1) ζ_1, β_1, θ_1; (2) β_2, ζ_2, θ_2; (3) ζ_3, θ_3, β_3; (4) β_4, θ_4, ζ_4; (5) θ_5, ζ_5, β_5; and (6) θ_6, β_6, ζ_6. The necessary changes of variable to demonstrate equivalence can be formulated from the torsion and then the elements of [T] can be shown to be equivalent for all six sequences. Alternatively, the change of variable can be formulated from the T_{23} element of [T] and then the torsion and bending curvatures can be shown to be equivalent for all six sequences. It is the second approach that is employed here.

We will make use of equations (5) extensively, which we repeat here for convenience

$$\left.\begin{aligned} (x + u)^+ &= T_{11} \\ v^+ &= T_{12} \\ w^+ &= T_{13} \end{aligned}\right\} \tag{A1}$$

These relationships hold for all possible sequences and types of rotations. The [T] matrix, the bending curvatures, and the torsion are now presented for each of the six sequences.

(1) ζ_1, β_1, θ_1.- The [T] matrix is

$$[T] = \begin{bmatrix} c_{\beta_1}c_{\zeta_1} & c_{\beta_1}s_{\zeta_1} & s_{\beta_1} \\ -c_{\theta_1}s_{\zeta_1} - s_{\theta_1}s_{\beta_1}c_{\zeta_1} & c_{\theta_1}c_{\zeta_1} - s_{\zeta_1}s_{\beta_1}s_{\theta_1} & c_{\beta_1}s_{\theta_1} \\ -c_{\theta_1}s_{\beta_1}c_{\zeta_1} + s_{\theta_1}s_{\zeta_1} & -s_{\theta_1}c_{\zeta_1} - s_{\zeta_1}s_{\beta_1}c_{\theta_1} & c_{\beta_1}c_{\theta_1} \end{bmatrix} \tag{A2}$$

The bending slopes are, from equation (A1)

$$\left.\begin{aligned} v^+ &= c_{\beta_1}s_{\zeta_1} \quad ; & v' &= \zeta_1 + O(\varepsilon^3) \\ w^+ &= s_{\beta_1} \quad ; & w' &= \beta_1 + O(\varepsilon^3) \end{aligned}\right\} \tag{A3}$$

The rotation rate vector is

$$\kappa = \zeta_1^+\mathbf{K} - \beta_1^+\left(c_{\zeta_1}\mathbf{J} - s_{\zeta_1}\mathbf{I}\right) + \theta_1^+\mathbf{i} = \kappa_i\mathbf{i} + \kappa_j\mathbf{j} + \kappa_k\mathbf{k} \tag{A4}$$

Thus

$$\left.\begin{aligned} \kappa_i &= \theta_1^+ + \zeta^+ s_{\beta_1} = \theta_1' + v''w' + O(\varepsilon^3) \\ \kappa_j &= -\beta_1^+ c_{\theta_1} + \zeta_1^+ c_{\beta_1} s_{\theta_1} = -w'' + \theta_1 v'' + O(\varepsilon^3) \\ \kappa_k &= \beta_1^+ s_{\theta_1} + \zeta_1^+ c_{\beta_1} c_{\theta_1} = v'' + \theta_1 w'' + O(\varepsilon^3) \end{aligned}\right\} \qquad \text{(A5)}$$

(2) β_2, ζ_2, θ_2.- The [T] matrix is

$$[T] = \begin{bmatrix} c_{\zeta_2} c_{\beta_2} & s_{\zeta_2} & c_{\zeta_2} s_{\beta_2} \\ -c_{\beta_2} c_{\theta_2} s_{\zeta_2} - s_{\beta_2} s_{\theta_2} & c_{\theta_2} c_{\zeta_2} & s_{\theta_2} c_{\beta_2} - c_{\theta_2} s_{\zeta_2} s_{\beta_2} \\ -s_{\beta_2} c_{\theta_2} + s_{\theta_2} s_{\zeta_2} c_{\beta_2} & -s_{\theta_2} c_{\zeta_2} & s_{\beta_2} s_{\theta_2} s_{\zeta_2} + c_{\beta_2} c_{\theta_2} \end{bmatrix} \qquad \text{(A6)}$$

The bending slopes are, from equations (A1)

$$\left.\begin{aligned} v^+ &= s_{\zeta_2} \quad ; \quad v' = \zeta_2 + O(\varepsilon^3) \\ w^+ &= c_{\zeta_2} s_{\beta_2} \quad ; \quad w' = \beta_2 + O(\varepsilon^3) \end{aligned}\right\} \qquad \text{(A7)}$$

The rotation rate vector is

$$\boldsymbol{\kappa} = -\beta_2^+ \mathbf{J} + \zeta_2^+ \left(c_{\beta_2} \mathbf{K} - s_{\beta_2} \mathbf{I} \right) + \theta_2^+ \mathbf{i} = \kappa_i \mathbf{i} + \kappa_j \mathbf{j} + \kappa_k \mathbf{k} \qquad \text{(A8)}$$

Thus

$$\left.\begin{aligned} \kappa_i &= \theta_2^+ - \beta_2^+ s_{\zeta_2} = \theta_2' - v'w'' + O(\varepsilon^3) \\ \kappa_j &= -\beta_2^+ c_{\theta_2} c_{\zeta_2} + \zeta_2^+ s_{\theta_2} = -w'' + \theta_2 v'' + O(\varepsilon^3) \\ \kappa_k &= \zeta_2^+ c_{\theta_2} + \beta_2^+ s_{\theta_2} c_{\zeta_2} = v'' + \theta_2 w'' + O(\varepsilon^3) \end{aligned}\right\} \qquad \text{(A9)}$$

(3) ζ_3, θ_3, β_3.- The [T] matrix is

$$[T] = \begin{bmatrix} c_{\beta_3}c_{\zeta_3} + s_{\beta_3}s_{\theta_3}s_{\zeta_3} & c_{\beta_3}s_{\zeta_3} - s_{\beta_3}s_{\theta_3}c_{\zeta_3} & c_{\theta_3}s_{\beta_3} \\ -c_{\theta_3}s_{\zeta_3} & c_{\theta_3}c_{\zeta_3} & s_{\theta_3} \\ -c_{\zeta_3}s_{\beta_3} + c_{\beta_3}s_{\theta_3}s_{\zeta_3} & -c_{\beta_3}s_{\theta_3}c_{\zeta_3} - s_{\beta_3}s_{\zeta_3} & c_{\beta_3}c_{\theta_3} \end{bmatrix} \tag{A10}$$

The bending slopes are, from equation (A1)

$$\left.\begin{aligned} v^+ &= c_{\beta_3}s_{\zeta_3} - s_{\beta_3}s_{\theta_3}c_{\zeta_3} \;; & v' &= \zeta_3 - \beta_3\theta_3 + O(\varepsilon^3) \\ w^+ &= c_{\theta_3}s_{\beta_3} \;; & w' &= \beta_3 + O(\varepsilon^3) \end{aligned}\right\} \tag{A11}$$

The rotation rate vector is

$$\boldsymbol{\kappa} = \zeta_3^+\mathbf{K} + \theta_3^+\left(c_{\zeta_3}\mathbf{I} + s_{\zeta_3}\mathbf{J}\right) - \beta_3^+\mathbf{j} = \kappa_i\mathbf{i} + \kappa_j\mathbf{j} + \kappa_k\mathbf{k} \tag{A12}$$

Thus

$$\left.\begin{aligned} \kappa_i &= \zeta_3^+c_{\theta_3}s_{\beta_3} + \theta_3^+c_{\beta_3} = \theta_3' + v''w' + O(\varepsilon^3) \\ \kappa_j &= -\beta_3^+ + \zeta_3^+s_{\theta_3} = -w'' + \theta_3v'' + O(\varepsilon^3) \\ \kappa_k &= \zeta_3^+c_{\beta_3}c_{\theta_3} - \theta_3^+s_{\beta_3} = v'' + \theta_3w'' + O(\varepsilon^3) \end{aligned}\right\} \tag{A13}$$

(4) β_4, θ_4, ζ_4.- The [T] matrix is

$$[T] = \begin{bmatrix} c_{\zeta_4}c_{\beta_4} - s_{\zeta_4}s_{\theta_4}s_{\beta_4} & s_{\zeta_4}c_{\theta_4} & c_{\zeta_4}s_{\beta_4} + s_{\zeta_4}s_{\theta_4}c_{\beta_4} \\ -s_{\zeta_4}c_{\beta_4} - s_{\beta_4}c_{\zeta_4}s_{\theta_4} & c_{\zeta_4}c_{\theta_4} & -s_{\beta_4}s_{\zeta_4} + c_{\zeta_4}s_{\theta_4}c_{\beta_4} \\ -c_{\theta_4}s_{\beta_4} & -s_{\theta_4} & c_{\beta_4}c_{\theta_4} \end{bmatrix} \tag{A14}$$

The bending slopes are, from (A1)

$$\left.\begin{aligned} v^+ &= s_{\zeta_4}c_{\theta_4} \;; & v' &= \zeta_4 + O(\varepsilon^3) \\ w^+ &= c_{\zeta_4}s_{\beta_4} + s_{\zeta_4}s_{\theta_4}c_{\beta_4} \;; & w' &= \beta_4 + \zeta_4\theta_4 + O(\varepsilon^3) \end{aligned}\right\} \tag{A15}$$

The rotation rate vector is

$$\boldsymbol{\kappa} = -\beta_4^+\mathbf{J} + \theta_4^+\left(c_{\beta_4}\mathbf{I} + s_{\beta_4}\mathbf{J}\right) + \zeta_4^+\mathbf{k} = \kappa_i\mathbf{i} + \kappa_j\mathbf{j} + \kappa_k\mathbf{k} \tag{A16}$$

Thus

$$\left.\begin{aligned} \kappa_i &= \theta_4^+ c_{\zeta_4} - \beta_4^+ s_{\zeta_4} c_{\theta_4} = \theta_4' - v'w'' + O(\varepsilon^3) \\ \kappa_j &= -\beta_4^+ c_{\zeta_4} c_{\theta_4} - \theta_4^+ s_{\zeta_4} = -w'' + \theta_4 v'' + O(\varepsilon^3) \\ \kappa_k &= \zeta_4^+ + \beta_4^+ s_{\theta_4} = v'' + \theta_4 w'' + O(\varepsilon^3) \end{aligned}\right\} \tag{A17}$$

(5) θ_5, ζ_5, β_5.- The [T] matrix is

$$[T] = \begin{bmatrix} c_{\beta_5}c_{\zeta_5} & c_{\beta_5}s_{\zeta_5}c_{\theta_5} - s_{\beta_5}s_{\theta_5} & c_{\beta_5}s_{\zeta_5}s_{\theta_5} + s_{\beta_5}c_{\theta_5} \\ -s_{\zeta_5} & c_{\zeta_5}c_{\theta_5} & c_{\zeta_5}s_{\theta_5} \\ -c_{\zeta_5}s_{\beta_5} & -c_{\beta_5}s_{\theta_5} - s_{\beta_5}s_{\zeta_5}s_{\theta_5} & c_{\beta_5}c_{\theta_5} - s_{\beta_5}s_{\zeta_5}s_{\theta_5} \end{bmatrix} \tag{A18}$$

The bending slopes are, from equations (A1)

$$\left.\begin{aligned} v^+ &= c_{\beta_5}s_{\zeta_5}c_{\theta_5} - s_{\beta_5}s_{\theta_5} = \zeta_5 - \beta_5\theta_5 + O(\varepsilon^3) \\ w^+ &= c_{\beta_5}s_{\zeta_5}s_{\theta_5} + s_{\beta_5}c_{\theta_5} = \beta_5 + \zeta_5\theta_5 + O(\varepsilon^3) \end{aligned}\right\} \tag{A19}$$

The rotation rate vector is

$$\boldsymbol{\kappa} = \theta_5^+\mathbf{I} + \zeta_5^+\left(c_{\theta_5}\mathbf{K} - s_{\theta_5}\mathbf{J}\right) - \beta_5^+\mathbf{j} = \kappa_i\mathbf{i} + \kappa_j\mathbf{j} + \kappa_k\mathbf{k} \tag{A20}$$

Thus

$$\left.\begin{aligned} \kappa_i &= \theta_5^+ c_{\beta_5}c_{\zeta_5} + \zeta_5^+ s_{\beta_5} = \theta_5' + v''w' + O(\varepsilon^3) \\ \kappa_j &= -\beta_5^+ - \theta_5^+ s_{\zeta_5} = -w'' + \theta_5 v'' + O(\varepsilon^3) \\ \kappa_k &= \zeta_5^+ c_{\beta_5} - \theta_5^+ c_{\zeta_5} s_{\beta_5} = v'' + \theta_5 w'' + O(\varepsilon^3) \end{aligned}\right\} \tag{A21}$$

(6) θ_6, β_6, ζ_6.- The [T] matrix is

$$[T] = \begin{bmatrix} c_{\zeta_6} c_{\beta_6} & s_{\zeta_6} c_{\theta_6} - c_{\zeta_6} s_{\beta_6} s_{\theta_6} & s_{\zeta_6} s_{\theta_6} + c_{\zeta_6} s_{\beta_6} c_{\theta_6} \\ -s_{\zeta_6} c_{\beta_6} & c_{\zeta_6} c_{\theta_6} + s_{\zeta_6} s_{\beta_6} s_{\theta_6} & c_{\zeta_6} s_{\theta_6} - s_{\zeta_6} s_{\beta_6} c_{\theta_6} \\ -s_{\beta_6} & -c_{\beta_6} s_{\theta_6} & c_{\beta_6} c_{\theta_6} \end{bmatrix} \tag{A22}$$

The bending slopes are, from equation (A1)

$$\left.\begin{aligned} v^+ &= s_{\zeta_6} c_{\theta_6} - c_{\zeta_6} s_{\beta_6} s_{\theta_6} \; ; \qquad v' = \zeta_6 - \beta_6 \theta_6 + O(\varepsilon^3) \\ w^+ &= s_{\zeta_6} s_{\theta_6} + c_{\zeta_6} s_{\beta_6} c_{\theta_6} \; ; \qquad w' = \beta_6 + \zeta_6 \theta_6 + O(\varepsilon^3) \end{aligned}\right\} \tag{A23}$$

The rotation rate vector is

$$\boldsymbol{\kappa} = \theta_6^+ \mathbf{I} - \beta_6^+ \left(c_{\theta_6} \mathbf{J} + s_{\theta_6} \mathbf{K} \right) + \zeta_6^+ \mathbf{k} = \kappa_i \mathbf{i} + \kappa_j \mathbf{j} + \kappa_k \mathbf{k} \tag{A24}$$

Thus

$$\left.\begin{aligned} \kappa_i &= \theta_6^+ c_{\zeta_6} c_{\beta_6} - \beta_6^+ s_{\zeta_6} = \theta_6' - v'w'' + (\varepsilon^3) \\ \kappa_j &= -\beta_6^+ c_{\zeta_6} - \theta_6^+ s_{\zeta_6} c_{\beta_6} = -w'' + \theta_6 v'' + O(\varepsilon^3) \\ \kappa_k &= \zeta_6^+ - \theta_6^+ s_{\beta_6} = v'' + \theta_6 w'' + O(\varepsilon^3) \end{aligned}\right\} \tag{A25}$$

Demonstration of Equivalence

It can be shown that changes of variables constructed from equating three elements of the [T] matrix for any two sequences will lead to identical expressions for bending curvatures and the torsion. Only one example is presented for illustration — the equivalence of (1) and (2). For convenience we choose to obtain the change of variable from T_{23}. From equations (A1), (A2), and (A6), the following relations hold:

$$\left.\begin{aligned} v^+ &= c_{\beta_1} s_{\zeta_1} = s_{\zeta_2} \\ w^+ &= s_{\beta_1} = c_{\zeta_2} s_{\beta_2} \\ T_{23} &= c_{\beta_1} s_{\theta_1} = s_{\theta_2} c_{\beta_2} - c_{\theta_2} s_{\zeta_2} s_{\beta_2} \end{aligned}\right\} \tag{A26}$$

Differentiation of equations (A26) yields

$$\left.\begin{aligned}
\zeta_1^+ c_{\beta_1} c_{\zeta_1} - \beta_1^+ s_{\beta_1} s_{\zeta_1} &= \zeta_2^+ c_{\zeta_2} \\
\beta_1^+ c_{\beta_1} &= \beta_2^+ c_{\zeta_2} c_{\beta_2} - \zeta_2^+ s_{\zeta_2} s_{\beta_2} \\
\theta_1^+ c_{\beta_1} c_{\theta_1} - \beta_1^+ s_{\beta_1} s_{\theta_1} &= \theta_2^+ \left(c_{\theta_2} c_{\beta_2} + s_{\theta_2} s_{\zeta_2} s_{\beta_2} \right) - \beta_2^+ \left(s_{\theta_2} s_{\beta_2} \right. \\
&\quad \left. + c_{\theta_2} s_{\zeta_2} c_{\beta_2} \right) - \zeta_2^+ c_{\theta_2} c_{\zeta_2} s_{\beta_2}
\end{aligned}\right\} \qquad \text{(A27)}$$

Thus

$$\left.\begin{aligned}
c_{\beta_1} &= \sqrt{1 - s_{\beta_1}^2} = \sqrt{1 - c_{\zeta_2}^2 s_{\beta_2}^2} \\
s_{\zeta_1} &= \frac{s_{\zeta_2}}{c_{\beta_1}} = \frac{s_{\zeta_2}}{\sqrt{1 - c_{\zeta_2}^2 s_{\beta_2}^2}} \\
c_{\zeta_1} &= \sqrt{1 - s_{\zeta_1}^2} = \frac{c_{\zeta_2} c_{\beta_2}}{\sqrt{1 - c_{\zeta_2}^2 s_{\beta_2}^2}} \\
s_{\theta_1} &= \frac{s_{\theta_2} c_{\beta_2} - c_{\theta_2} s_{\zeta_2} s_{\beta_2}}{\sqrt{1 - c_{\zeta_2}^2 s_{\beta_2}^2}} \\
c_{\theta_1} &= \sqrt{1 - s_{\theta_1}^2} = \frac{c_{\theta_2} c_{\beta_2} + s_{\theta_2} s_{\zeta_2} s_{\beta_2}}{\sqrt{1 - c_{\zeta_2}^2 s_{\beta_2}^2}}
\end{aligned}\right\} \qquad \text{(A28)}$$

Substitution of equations (A28) into (A27) yields

$$\beta_1^+ = \frac{\beta_2^+ c_{\zeta_2} c_{\beta_2} - \zeta_2^+ s_{\zeta_2} s_{\beta_2}}{\sqrt{1 - c_{\zeta_2}^2 s_{\beta_2}^2}}$$

$$\zeta_1^+ = \frac{\zeta_2^+}{c_{\beta_2}} + \frac{s_{\beta_2} s_{\zeta_2}}{c_{\beta_2}} \left(\frac{\beta_2^+ c_{\zeta_2} c_{\beta_2} - \zeta_2^+ s_{\zeta_2} s_{\beta_2}}{1 - c_{\zeta_2}^2 s_{\beta_2}^2} \right)$$

$$\theta_1^+ = \theta_2^+ - \frac{\beta_2^+ s_{\zeta_2} + \zeta_2^+ s_{\beta_2} c_{\zeta_2} c_{\beta_2}}{1 - c_{\zeta_2}^2 s_{\beta_2}^2}$$

Substitution of equations (A27) through (A29) into equations (A5) yields for κ_i

$$\kappa_i = \theta_2^+ - \frac{\beta_2^+ s_{\zeta_2} + \zeta_2^+ s_{\beta_2} c_{\zeta_2} c_{\beta_2}}{1 - c_{\zeta_2}^2 s_{\beta_2}^2} + \frac{s_{\beta_2} c_{\zeta_2}}{c_{\beta_2}} \left[\zeta_2^+ + s_{\beta_2} s_{\zeta_2} \left(\frac{\beta_2^+ c_{\zeta_2} c_{\beta_2} - \zeta_2^+ s_{\zeta_2} s_{\beta_2}}{1 - c_{\zeta_2}^2 s_{\beta_2}^2} \right) \right]$$

$$= \theta_2^+ - s_{\zeta_2} \beta_2^+ \qquad \text{(A29)}$$

which is identical to the expression for κ_i in equations (A9). The bending curvatures also transform identically. Similarly, the other four sequences (3)-(6) can be shown to be equivalent to (1). The changes of variable to second order may be determined by inspection, which proves that all six sequences are also equivalent to second order.

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15. Supplementary Notes

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16. Abstract Nonlinear expressions are developed to relate the orientation of the deformed-beam cross section, torsion, local components of bending curvature, angular velocity, and virtual rotation to deformation variables. These expressions are developed in an exact manner in terms of a quasi-coordinate in the space domain for the torsion variable. The entire formulation is independent of the sequence of the three rotations used to describe the orientation of the deformed-beam cross section. For more common cases in the literature in which one of the three rotation angles is used as the torsion variable, the resulting equations depend on the choice of the three angles. Differences in the equations, however, are demonstrated to be in form only. The present deformed-beam kinematic quantities are proven to be equivalent to those derived from various rotation sequences by identifying appropriate changes of variable based on fundamental uniqueness properties of the deformed beam geometry. This development helps to clarify the issues raised in the literature concerning the choice of the angles. The torsion variable used herein is also shown to be mathematically analogous to an axial deflection variable that has been commonly used in the literature. Both variables are quasi-coordinates in the space domain and have been used in derivations based on Hamilton's principle, despite lack of rigorous justification. Rigorous applicability of Hamilton's principle to systems described by a class of quasi-coordinates that includes these variables is formally established.

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